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SUBJECT: MULTIVARIATE CALCULUS

INSTRUCTOR: SIR DAUD

Q1:- Let $P(1,0,-3)$, $Q(0,-2,-4)$ and $R(4,1,6)$ be points.

a) Find the equation of the plane through points P, Q, R.

Solution:-

Since equation of plane through given points is a $\cancel{a(x-x_0)+b(y-y_0)}$

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0 \quad \text{—(i)}$$

where (x_0, y_0, z_0) is any point on plane and $\langle a,b,c \rangle$ is perpendicular to the plane.

Thus $\overrightarrow{PQ} = \langle 4-1,\ 1-0,\ 6+3 \rangle$

$= \langle 3, 1, 9 \rangle$

Now we take cross product of the above two vectors

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & -1 \\ 3 & 1 & 9 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -2 & -1 \\ 1 & 9 \end{vmatrix} - \hat{j}\begin{vmatrix} -1 & -1 \\ 3 & 9 \end{vmatrix} + \hat{k}\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix}$$

$$= \hat{i}(-18+1) - \hat{j}(-9+3) + \hat{k}(-1+6)$$

$$= \hat{i}(-17) - \hat{j}(-6) + \hat{k}(5)$$

$$= -17\hat{i} + 6\hat{j} + 5\hat{k}$$

be

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let P be any point on plane,
so putting values in eq(1)

$$-17(x-1)+(6)(y-0)+5(z-(-3))=0$$

$$-17(x-1)+6(y)+5(z+3)=0$$

$$-17x+17+6y+5z+15=0$$

$$-17x+6y+5z+32=0$$

$$-17x+6y+5z=-32$$

Multiplying both sides by (-1)

$$17x-6y+5z=32$$

which is required equation of plane through given three points.

b) Find the area of triangle with vertices P, Q & R.

Solution:-

Since area of triangle of the given vertices is

$$\text{area of } \Delta = \frac{|\vec{PQ}\times\vec{PR}|}{2} \quad \text{—(i)}$$

As $\vec{PQ} = \langle -1,-2,-1 \rangle$ and $\vec{PR} = \langle 3,1,9 \rangle$

their cross product is

$|\vec{PQ} \times \vec{PR}| = \langle -17, 6, 5 \rangle$

putting values in eq (i)

$$\text{area of } \Delta = \frac{\sqrt{(-17)^2 + (6)^2 + (5)^2}}{2}$$

$$= \sqrt{289 + 36 + 25} / 2$$

$$= \sqrt{350} / 2$$

area of triangle $\Delta = \frac{5\sqrt{14}}{2}$ units

Q2: Let $f(x,y) = (x-y)^3 + 2xy + x^2 - y$.
Find the linear approximation $L(x,y)$
near the point $(1,2)$

Solution:-
Linear approximation $L(x,y)$ near the given point is

$$L(x,y) = f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0) \quad \text{-(i)}$$

As given $f(x,y) = (x-y)^3 + 2xy + x^2 - y$

and $P(x_0,y_0) = P(1,2)$

$f(x,y) = (x-y)^3 + 2xy + x^2 - y$

$f(1,2) = (1-2)^3 + 2(1)(2) + (1)^2 - 2$

$= (-1)^3 + 4 + 1 - 2 = \cancel{-1} + 4 + \cancel{1} - 2$

$= 2$

Now we find derivatives w.r.t x & y

$\cancel{f_x}\ f_x(x,y) = 3(x-y)^2 + 2y + 2x$

$f_x(1,2) = 3(1-2)^2 + 2(2) + 2(1)$

$3(-1)^2 + 4 + 2 = +3 + 4 + 2 = 7 + 2 = 9$

and $f_y(x,y) = 3(x-y)^2(-1) + 2x - 1$

$\cancel{f_y(x,y)}\ f_y(1,2) = 3(1-2)^2(-1) + 2(1) - 1$

$= 3(-1)^2(-1) + 2 - 1 = -3 + 2 - 1 = -2$

putting values in eq (i)

$L(x,y) = f(1,2) + f_x(x-1) + f_y(y-2)$

Let (x,y) be $(1.1, 1.9)$, then

$= 2 + 9(x-1) + (-2)(y-2)$

$= 2 + 9(1.1-1) - 2(1.9-2)$

$= 2 + 9(0.1) - 2(-0.1)$

$= \cancel{2+9}\ 2 + 0.9 + 0.2$

$= 3.1$ Answer

Q3: Find the distance between parallel planes $x+2y-z=-1$ and $3x+6y-3z=3$.

Solution:-

Given that

$$x+2y-z=1 \quad \text{—(i)}$$
$$3x+6y-3z=3 \quad \text{—(ii)}$$

are two parallel planes. Formula for distance between parallel planes is

$$d=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Put $y=z=0$ in eq(i), we get $x=-1$

Thus $P(x_1,y_1,z_1)=P(-1,0,0)$ is a point on plane (i). Now we find perpendicular distance from point $P(-1,0,0)$ to plane (ii)

$$d=\frac{|3(-1)+6(0)+(3)(0)+(-3)|}{\sqrt{(3)^2+(6)^2+(-3)^2}} \cdot \frac{|-3-3|}{\sqrt{9+36+9}}$$

$$=\frac{|-6|}{\sqrt{54}} \qquad =\frac{6}{\sqrt{54}}$$

Q4: Find the following limit, if it exist, or show that the limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy+y^2}{x^2+y^2}$$

Solution:-

Given that

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy+y^2}{x^2+y^2}$$

$\boxed{x\text{-axis}}$ limit along the path $y=0$

$$= \lim_{(x,y)\to(0,0)} \left.\frac{x^2-xy+y^2}{x^2+y^2}\right|_{y=0}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2-x(0)+(0)^2}{x^2-(0)^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$$

$\boxed{y\text{ axis}}$ limit along the path $x=0$

$$= \lim_{(x,y)\to(0,0)} \left.\frac{x^2-xy+y^2}{x^2+y^2}\right|_{x=0}$$

$$= \lim_{(0,y)\to(0,0)} \frac{(0)^2-(0)(y)+(y)^2}{(0)^2+(y)^2} = \lim_{(0,y)\to(0,0)} \frac{y^2}{y^2} = 1$$

$$= \lim_{(x,x)\to(0,0)} \frac{x^2 - x\cdot x + x^2}{x^2 + x^2} = \lim_{(x,x)\to(0,0)} \frac{\cancel{x^2} - \cancel{x^2} + x^2}{2x^2}$$

$$= \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$$

As we obtained limits, so the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy + y^2}{x^2+y^2}$$ does not exist.

Q5:- Find the directional derivative of the function $f(x,y,z) = xyz$ in the direction of vector $v = <5,-3,2>$

Solution:-

Given function $f(x,y,z) = xyz$

and vector $v = <5,-3,2>$

Since directional derivative of the function is

$$DD = \nabla f \cdot \hat{v} \text{ —— (i)}$$

So just we will find ∇f and $\hat{v}$

$$\nabla f = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial j}\hat{j} + \frac{\partial}{\partial k}\hat{k}$$

$= yz\hat{i} + xz\hat{j} + xy\hat{k}$

Now,

$$\hat{v} = \frac{V}{|V|} = \frac{5\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{5^2 + (-3)^2 + (2)^2}} = \frac{5i - 3j + 2k}{\sqrt{25 + 9 + 4}}$$

$$= \frac{5i - 3j + 2k}{\sqrt{38}}$$

putting values in (i)

$$DD = (yz\hat{i} + xz\hat{j} + xy\hat{k}) \frac{(5i - 3j + 2k)}{\sqrt{38}} = \frac{5yz - 3xz + 2xy}{\sqrt{38}}$$

ANS.

Q6:- Find the equation of tangent plane to the surface $z = 4x^3y^2 + 2y$ at point $(1, -2, 12)$.

Solution:-

Given that

$$z = 4x^3y^2 + 2y$$

or

$$4x^3y^2 + 2y - z = 0$$

Since equation of tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is is $F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)$

$+F_z(x_0,y_0,z_0)(z-z_0)=0$ —— (a)

Thus $F(x,y,z)=4x^3y^2+2y-z=0$

$F_z=12x^2y^2$

$F_x(1,-2,12)=12(1)^2(-2)^2=12(1)(4)=48$

So $F_y=8x^3y+2$

$F_y(1,-2,12)=8(1)^3(-2)+2=8(-2)+2=-16+2=-14$

$F_z=-1$

Now putting values in (a)

$48(x-1)+(-14)(y-(-2))+(-1)(z-12)=0$

$48(x-1)-14(y+2)-1(z-12)=0$

$48x-48-14y-28-z+12=0$

$48x-14y-z-64=0$

or

$\boxed{48x-14y-z=64}$ Ans

Q:7:- Let $u = \langle u_1, u_2, u_3 \rangle$ & $v = \langle v_1, v_2, v_3 \rangle$ be any two vectors in space. Show the following identity that relates the cross product and the dot product $|u \times v|^2 + |u.v|^2 = |u|^2 |v|^2$

Solution:-

Given two vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = |v_1, v_2, v_3|$ and if θ is an angle between u and v. So

$$|u \times v| = |\bar{u}| |\bar{v}| \sin\theta \, \hat{n}$$
$$|u.v| = |\bar{u}| |\bar{v}| \cos\theta$$

Since we have

$$|u \times v|^2 + |u.v|^2 = |u|^2 |v|^2$$

So

$$= (|\bar{u}| |\bar{v}| \sin\theta)^2 + (|\bar{u}| |\bar{v}| \cos\theta)^2$$

$$= |u|^2 |v|^2 \sin^2\theta + |u|^2 |v|^2 \cos^2\theta$$

$$= |u|^2 |v|^2 \sin^2\theta + |u|^2 |v|^2 \cos^2\theta$$

$$= |u|^2 |v|^2 (\sin^2\theta + \cos^2\theta)$$

$$= |u|^2 |v|^2 (1) \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= |u|^2 |v|^2$$

Q8:- What is the angle between the two ~~vectors~~ planes $x+y=0$ & $y-z=2$?

Solution:-

Since angle between planes with normals is

$$\cos Q = \frac{|\vec{n_1}\cdot\vec{n_2}|}{||\vec{n_1}||\,||\vec{n_2}||} \longrightarrow (a)$$

when n_1 and n_2 are normals vectors. so,

$$x+y=0 \quad \text{—(i)}$$
$$y-z=0 \quad \text{—(ii)}$$

Thus

$$\vec{n_1} = \langle 1,1,0\rangle \text{ and } \vec{n_2} = \langle 0,1,-1\rangle$$

$$\vec{n_1}\cdot\vec{n_2} = (1\times 0) + (1\times 1) + (0\times -1)$$
$$= 0+1+0 = 1$$

$$||\vec{n_1}|| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$$

$$||\vec{n_2} = \sqrt{0^2+1^2+\cancel{0}(-1)^2} = \sqrt{2}$$

putting values in (a)

$$\cos Q = \frac{|1|}{(\sqrt{2})(\sqrt{2})} = \frac{1}{2}$$

$$Q = \cos^{-1}(1/2), \quad Q = 60^\circ$$

THE END.