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Subject # Differential Equation

Q.No:1

Solution:

$$\frac{dy}{dt} = e^{y-t}\sec y\,(1+t^2)$$

$$\frac{dy}{dt} = \frac{e^{y}}{e^{t}}\,\frac{1}{\cos y}(1+t^2)$$

or $\quad e^{-y}\cos y\,dy = e^{-t}(1+t^2)\,dt$

Integrating b/s

$$\int e^{-y}\cos y\,dy = \int e^{-t}(1+t^2)\,dt \quad \text{— (i)}$$

By parts

$$\cos y\cdot\frac{e^{-y}}{-1} - \int -\sin y\cdot\frac{e^{-y}}{-1}\,dy = \int e^{-t}(1+t^2)\,dt$$

$$-e^{-y}\cos y - \left[-\sin y\cdot e^{-y} + \int \cos y\cdot e^{-y}\,dy\right] = \int e^{-t}(1+t^2)\,dt$$

$$-e^{-y}\cos y + e^{-y}\sin y - \int \cos y\cdot e^{-y}\,dy = \int e^{-t}(1+t^2)\,dt$$

$$e^{-y}(\sin y - \cos y) - \int \cos y\cdot e^{-y}\,dy = \int e^{-t}(1+t^2)\,dt$$

From Equation (i)

$$\int e^{-t}(1+t^2)\,dt = \int e^{-y}\cos y\,dy$$

So $\quad e^{-y}(\sin y - \cos y) - \int e^{-t}(1+t^2)\,dt = \int e^{-t}(1+t^2)\,dt$

(v)

$$e^{-y}(\sin y-\cos y) = \int e^{-t}(1+t^2)\,dt + \int e^{-t}(1+t^2)\,dt$$

$$e^{-y}(\sin y-\cos y) = 2\int e^{-t}(1+t^2)\,dt.$$

$$'' \quad = 2\left[(1+t^2)\frac{e^{-t}}{-1} + \int 2t\cdot e^{-t}\,dt\right.$$

$$e^{-y}(\sin y-\cos y) = 2\left[e^{-t}(1+t^2)+2\left(\frac{te^{-t}}{-1}+\int e^{-t}\,dt\right)\right]$$

$$'' \quad = 2\left[e^{-t}(1+t^2)+2[te^{-t}-e^{-t}\right]+C$$

$$e^{-y}(\sin y-\cos y) = 2\left[-e^{-t}-e^{-t}\cdot t^2-2t\cdot e^{-t}-2e^{-t}\right]+C$$

$$e^{-y}(\sin y-\cos y) = 2\left[-3e^{-t}-t^2e^{-t}-2t\cdot e^{-t}\right]+C$$

$$\boxed{e^{-y}(\sin y-\cos y) = -2e^{-t}\left[t^2+2t+3\right]+C}$$

Q.No:2

Solution:

Q2 $\cdot\ (\sqrt{x+y}+\sqrt{x-y})\,dx-(\sqrt{x+y}-\sqrt{x-y})\,dy=0$

Sol Above Equation becomes.

$$\sqrt{x+y}+\sqrt{x-y}\ dx=\sqrt{x+y}-\sqrt{x-y}\ dy$$

or $$dy/dx=\frac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}$$

Let $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\Rightarrow\ v+x\frac{dv}{dx}=\frac{\sqrt{x+vx}+\sqrt{x-vx}}{\sqrt{x+vx}-\sqrt{x-vx}}$$

$$=\frac{\sqrt{x(1+v)}+\sqrt{x(1-v)}}{\sqrt{x(1+v)}-\sqrt{x(1-v)}}$$

$$=\frac{\sqrt{x}.\sqrt{1+v}+\sqrt{x}(\sqrt{1-v})}{\sqrt{x}.\sqrt{1+v}-\sqrt{x}.\sqrt{1-v})}$$

$$=\frac{\sqrt{x}(\sqrt{1+v}+\sqrt{1-v})}{\sqrt{x}(\sqrt{1+v}-\sqrt{1-v})}=\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}$$

$\div$ing and $\times$ing R.H.S by conjugate of Denominator.

$$\Rightarrow\ v+x\frac{dv}{dx}=\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}\cdot\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}$$

$$v + x\frac{dv}{dx} = \frac{\left(\sqrt{1+v} + \sqrt{1-v}\right)^2}{\left(\sqrt{1+v}\right)^2 - \left(\sqrt{1-v}\right)^2}$$

$$= \frac{\left(\sqrt{1+v}\right)^2 + \left(\sqrt{1-v}\right)^2 + 2\left(\sqrt{1+v}\right)\left(\sqrt{1-v}\right)}{1+v-(1-v}$$

$$= \frac{1+\cancel{v}+1-\cancel{v} + 2\sqrt{(1+v)(1-v)}}{1-1+v+v}$$

$$= \frac{2 + 2\sqrt{(1+v)(1-v)}}{2v}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{2}\left(1 + \sqrt{1-v^2}\right)}{\cancel{2}v}$$

$$x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}}{v} - v$$

$$x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2} - v^2}{v} = \frac{1-v^2 + \sqrt{1-v^2}}{v}$$

$$x\frac{dv}{dx} = \frac{\sqrt{1-v^2}\left(1 + \sqrt{1-v^2}\right)}{v}$$

Separating variables

$$\frac{v\,dv}{\sqrt{1-v^2}\,(1+\sqrt{1-v^2})} = \frac{dx}{x}$$

Let $1+\sqrt{1-v^2} = t$

$$\Rightarrow dt = \frac{1}{\cancel{2}}(1-v^2)^{-1/2} \cdot \cancel{2}v\,dv$$

$$dt = -\frac{v}{\sqrt{1-v^2}}\,dv$$

Substituting in above Equation

$$\Rightarrow \int -\frac{dt}{t} = \frac{dx}{x}$$

integrating b/s

$$\int -\frac{dt}{t} = \int \frac{dx}{x}$$

$$-\ln t = \ln x + \ln c$$

$$-\ln t = \ln cx \quad \text{or} \quad \ln t = -\ln cx$$

$$\ln t = \ln(cx^{-1})$$

$$\Rightarrow cx^{-1} = t$$

$$\Rightarrow t = 1/cx$$

Reputting value of t

$$1+\sqrt{1-v^2} = 1/cx$$

$$1 + \sqrt{1 - y^2/x^2} = \frac{1}{cx}$$

$$1 + \sqrt{\frac{x^2 - y^2}{x^2}} = \frac{1}{cx}$$

$$1 + \frac{\sqrt{x^2 - y^2}}{x} = 1/cx$$

$$\frac{x + \sqrt{x^2 - y^2}}{x} = \frac{1}{cx}$$

$$x + \sqrt{x^2 - y^2} = \frac{1}{c}$$

or $$\boxed{C = x + \sqrt{x^2 - y^2}}$$

Q.No:3

Solution:

Q.No 3 $(D^4+D^2)y = 3x^2+4\sin x-2\cos x$ ①

Sol $(D^4+D^2)y = 3x^2+4\sin x-2\cos x$

$\Rightarrow \quad f(D)y = f(x)$

As it is non-homogenous linear equation

so solution will be

$$y = y_c + y_p \quad \text{---(i)}$$

Complementary Solution y_c

$$D^4+D^2 = 0 \Rightarrow D^2(D^2+1) = 0$$

Either $D^2 = 0 \Rightarrow \boxed{D=0}$.

$D^2+1 = 0 \Rightarrow D^2 = -1$

$D = \sqrt{-1} \Rightarrow \boxed{D=i}$. or $D = \boxed{0 \pm i}$

Roots are real and complex

$$y_c = c_1e^{0x} + e^{0x}(c_2\cos x + c_3\sin x)$$

$$y_c = c_1 + c_2\cos x + c_3\sin x$$

$$y_p = \frac{1}{f(D)}F(x)$$

$$y_p = \frac{1}{D^4+D^2}\left(3x^2+4\sin x-2\cos x\right)$$

$$= \frac{3x^2}{D^4+D^2} + \frac{4\sin x}{D^4+D^2} - \frac{2\cos x}{D^4+D^2}.$$

$$f(D) = D^4+D^2$$

at $D=0 \Rightarrow f(D)=0$

So $f'(D) = 4D^3+2D$

Now also for $D=0 \Rightarrow f'(0) = 0$.

again differentiating.

$$f''(D) = 12D+2$$

So for $D=0$

$$f''(0) = 12(0)+2 = 2.$$

So replacing $\frac{1}{\cancel{D^2} f(D)}$ with $\frac{x^2}{f''(D)}$

$$\Rightarrow y_p = \frac{x^2\,3x^2}{12D+2} + \frac{x^2}{12D+2}\cdot 4\sin x - \frac{x^2}{12D+2}\cdot 2\cos x$$

So putting $D=0$ in all

$$y_p = \frac{x^2 \cdot 3x^2}{12(0)+2} + \frac{x^2 \cdot 4\sin x}{12(0)+2} - \frac{2x^2\cos x}{12(0)+2}$$

$$y_p = \frac{3x^4}{2} + \frac{4x^2\sin x}{2} - \frac{2x^2\cos x}{2}$$

$$= \frac{3}{2}x^4 + 2x^2\sin x - x^2\cos x$$

So putting in Equation (1')

$$y = C_1 + C_2\cos x + C_3\cos x + \frac{3}{2}x^4 + 2x^2\sin x - x^2\cos x$$

$$\boxed{y = C_1 + (C_2 - x^2)\cos x + (C_3 + 2x^2)\sin x + \frac{3}{2}x^4}$$