FINAL EXAM

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SUBJECT :- DIFFERENTIAL EQUATIONS

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Question :- 01 (A)

Show the following functions are all Solutions of the wave equation by determining relevant partial derivates.

Given that :-

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

(i) $w = \sin(x + ct) + \cos(2x + 2ct)$
(ii) $w = \tan(2x + ct)$

Required that :-

We have to show that these two are Soltion of wave equation.

Solution :-

(i) $w = \sin(x + ct) + \cos(2x + ct)$

Taking partial derivative with respect to "t".

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\{\sin(x+ct)\} + \frac{\partial}{\partial t}\{c\cos(2x+ct)\}$$

$$\frac{\partial w}{\partial t} = \cos(x+ct) \cdot c + (-\sin(x+ct))$$

$$\frac{\partial w}{\partial t} = c\{\cos(x+ct) - 2c\sin(2x+2ct)\}$$

$$\frac{\partial w}{\partial t} = c\{\cos(x+ct) - 2\sin(2x+2ct)\}$$

Again Taking Derivative

$$\frac{\partial^2 w}{\partial t^2} = c\{\frac{\partial}{\partial t}\cos(x+ct) - 2\frac{\partial}{\partial t}\sin(2x+2ct)\}$$

$$\frac{\partial^2 w}{\partial t^2} = c\{(-\sin(x+ct) \cdot c - 2\cos(2x+2ct)) \cdot 2c\}$$

$$\frac{\partial^2 w}{\partial t} = c\{-c\sin(x+ct) - 4c\cos(2x+2ct)\}$$

$$\frac{\partial^2 w}{\partial t} = c^2\{-\sin(x+ct) - 4\cos(2x+2ct)\}$$

Now,

$$w = \sin(x+ct) + \cos(2x+2ct)$$

Taking derivative w.r.t x

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\sin(x+ct) + \frac{\partial}{\partial x}\cos(2x+2ct)$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) + (-\sin(2x+2ct)\cdot 2)$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - 2\sin(2x+2ct)$$

Now Taking Again Derivative

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\cos(x+ct) - 2\frac{\partial}{\partial x}\sin(2x+2ct)$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct)\cdot 1 - 2\cos(2x+2ct)\cdot 2$$

$$\frac{\partial^2 w}{\partial w^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

Now wave equation is,

$$\frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial \omega}{\partial x}$$

putting values

$$\Rightarrow c^2\{-\sin(x+ct) - \cancel{4}\cos(2x+2ct)\}$$

$$= c^2\{-\sin(x+ct) - \cancel{4}\cos(2x+2ct)\}$$

$$\Rightarrow 1 = 1$$

L.H.S = R.H.S

Result :-

Hence It is the solution of wave equation.

(ii) **Question :- 01 part (B)**

$w = \tan(ax+ct)$

Taking Derivative w.r.t t, then

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t} \tan(ax+ct)$$

$$\frac{\partial w}{\partial t} = \sec^2(ax+ct) \cdot c$$

$$\frac{\partial w}{\partial t} = c \cdot \sec^2(ax+ct)$$

Taking again Derivative

$$\frac{\partial^2 w}{\partial t^2} = c \frac{\partial}{\partial t} \sec^2(ax+ct)$$

$$\frac{\partial^2 w}{\partial t^2} = c \cdot 2\sec(ax+ct) \cdot \sec(ax+ct) \cdot \tan(ax+ct)$$

$$\frac{\partial^2 w}{\partial t^2} = 2c^2 \sec^2(ax+ct) \cdot \tan(ax+ct)$$

Now, $w = \tan(ax+ct)$

Taking derivative w.r.t to x,

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \tan(2x+ct)$$

$$\frac{\partial w}{\partial x} = \sec^2(2x+ct) \cdot 2$$

$$\frac{\partial w}{\partial x} = 2\sec^2(2x+ct)$$

Taking Again Derivative.

$$\frac{\partial^2 w}{\partial x^2} = 2 \cdot \frac{\partial}{\partial x} \sec^2(2x+ct)$$

$$\frac{\partial^2 w}{\partial x^2} = 2 \cdot 2 \sec(2x+ct) \sec(2x+ct) \cdot \tan(2x+ct) \cdot 2$$

$$\frac{\partial^2 w}{\partial x^2} = 6 \sec^2(2x+ct) \tan(2x+ct)$$

Now wave equation is,

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

putting values

$$\cancel{2}^{1}\cancel{c^2}\sec^2(2x+ct)\tan(2x+ct)$$

$$= \cancel{c^2}\,\cancel{6}^{3}\sec^2(2x+ct)\tan(2x+ct)$$

$$\Rightarrow \quad 1 \neq 3$$

L.H.S $\neq$ R.H.S

Result :-

Hence It is not the Solution of the wave equation.

Question :- 02

Expand the function in a fourier Series,

$f(x) = x, \quad -\pi < x \leq 0$

$= 2x \quad 0 \leq x \leq \pi.$

Given Data :-

$f(x) = x \quad , \quad -\pi \leq x \leq 0$

$= 2x \quad , \quad 0 \leq x \leq \pi$

Required Data :-

we have to find,

to expand the function in a fourier Series.

Solution :-

Here we have formula

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

Here

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 2x\,dx$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\left|\frac{(x^2)}{2}\right|_{-\pi}^{0} + \frac{1}{\cancel{2}\pi}(\cancel{2})\left|\frac{x^2}{\cancel{2}}\right|_{0}^{\pi}$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\left[0-\frac{\pi^2}{2}\right] + \frac{1}{2\pi}\left[\pi^2-0\right]$$

$$\Rightarrow a_0 = \frac{\cancel{\pi^2}}{4\cancel{\pi}} + \frac{1}{2\cancel{\pi}}(\cancel{\pi^2})$$

$$a_0 = \frac{1}{4}\pi + \frac{1}{2}\pi$$

Now,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left\{x\frac{\sin nx}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\left(\frac{dx}{dx}\left(\frac{\sin(nx)}{n}\right)dx\right\}\right.$$

$$+\frac{2}{\pi}\left\{x\frac{\sin nx}{n}\Big|_{0}^{\pi} - \int_{0}^{\pi}\left(\frac{d(x)}{dx}\cdot\frac{\sin nx}{n}\,dx\right)\right\}$$

$$a_n = \frac{1}{\pi}\left(\frac{\cos nx}{n^2}\Big|_{-\pi}^{0}\right) + \frac{2}{\pi}\left(\frac{\cos nx}{n^2}\Big|_{0}^{\pi}\right)$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}\left(\cos n(0) - \cos n(-\pi)\right) + \frac{2}{n^2\pi}\left(\cos n\pi - \cos n(0)\right)$$

$$\Rightarrow a_n = \frac{1}{n^2\pi}(1+1) + \frac{2}{n^2\pi}(-1-1)$$

$$\Rightarrow a_n = \frac{2}{n^2\pi} + \left(\frac{-4}{n^2\pi}\right)$$

$$\Rightarrow a_n = \frac{-2}{n^2\pi}$$

Now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\left(\frac{d}{dx}x\left(-\frac{\cos nx}{n}\right)\right)dx\right.$$

$$\Rightarrow + \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)\Big|_0^{\pi} - \int_0^{\pi}\left[\frac{d}{dx}x\left(-\frac{\cos nx}{n}\right)dx\right]\right]$$

$$= b_n = \frac{1}{\pi}\left(\frac{x n(2)}{n}\int_{-\pi}^{0}\left(-\frac{\cos nx}{n}\right)dx\right)$$

$$+ \frac{2}{\pi}\left(\frac{2x\cancel{\pi}}{\cancel{\pi}} - \int_0^{\pi} -\frac{\cos nx\,dx}{\pi}\right)$$

$$b_n = \frac{1}{\pi} - 2x - \left[-\cancel{\frac{\sin nx}{n^2}}\Big|_{-\pi}^{0}\right] + \frac{4x}{\pi} - \left(-\cancel{\frac{\sin nx}{n^2}}\Big|_0^{-\pi}\right]$$

$$b_n = \frac{-2x}{\pi} + \frac{4x}{\pi}$$

$$b_n = \frac{2x}{\pi}$$

Now putting values in eq (1)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\left| F(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty} -\frac{2}{n^2\pi}\cos nx + \sum_{n=1}^{\infty}\frac{2x}{\pi}\sin nx \right|$$

Hence It is the Required Series.

Question :- 03

Solve the initial value problem,

$y'' - 4y' + 13y = 8\sin 3x$, $y(0) = 1$ and $y'(0) = 2$

Solution :-

$y'' - 4y' + 13y = 8\sin 3x$

So, $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 8\sin 3x$

$y\left(\frac{d^2}{dx^2} - 4\frac{d}{dx} + 13\right) = 8\sin 3x$

Now, put $\frac{d}{dx} = D$

$y(D^2 - 4D + 13) = 8\sin 3x$

put $D = \Delta$

$y(\Delta^2 - 4D + 13) = 8\sin 3x$

Now, characteristic equation is,

$$\Delta^2 - 4\Delta + 13 = 0$$

$a = 1,\ b = -4,\ c = 13$

Now from Quadratic Equation

$$\Delta = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$\Delta = \frac{4 \pm \sqrt{16-52}}{2}$$

$$\Delta = \frac{4 \pm \sqrt{-36}}{2}$$

$$\Delta = \frac{4 \pm \sqrt{36}\, i}{2}$$

$$\Delta = \frac{4 \pm 6i}{2}$$

$$\Delta = 2 \pm 3i$$

Now

$m_1 = 2 + 3i$

$m_2 = 2 - 3i$

Now $y_c = e^{2x}\{C_1 \cos 3x + C_2 \sin 3x\}$

Now,

$$y_p = \text{Img}\left(\frac{1}{m^2-4m+13}\, 8e^{3ix}\right)$$

$$y_p = 8\,\text{Img}\ \frac{e^{3ix}}{(3i)^2-4(3i)+13}$$

$$y_p = 8\,\text{Img}\ \frac{e^{3ix}}{9i^2-12i+13}$$

$$y_p = 8\,\text{Img}\ \frac{e^{3ix}}{4-12i}$$

$$y_p = \cancel{8}^{2}\,\text{Img}\ \frac{e^{3ix}}{\cancel{4}(1-3i)}$$

$$y_p = 2\,\text{Img}\ \frac{e^{3ix}}{1-3i}$$

$$y_p = 2\,\text{Img}\ \frac{e^{3ix}}{1-3i} \times \frac{1+3i}{1+3i}$$

$$y_p = \text{Img}\frac{(1+3i)e^{3ix}}{(1)^2-(3i)^2}$$

$$y_p = 2\,\text{Img}\frac{(1+3i)e^{3ix}}{10}$$

$$y_p = \frac{2}{10}(1+3i)(\cos 3x + 5\sin 3x)$$

$$y_p = \frac{1}{5}(\sin 3x + 3\cos 3x)$$

So the general solution is

$$y = y_c + y_p$$

$$y = e^{2x}\{C_1 \cos 3x + C_2 \sin 3x\} + \frac{1}{5}(\sin 3x + 3\cos 3x)$$

Now $y(0) = 1$

$$y(0) = e^{2(0)}\{C_1 \cos 3(0) + C_2 \sin 3(0)\}$$
$$+ \frac{1}{5}(\sin 3(0) + 3\cos(0))$$

$$\Rightarrow 1 = 1(c_1 + 0) + \frac{1}{5}(3)$$

$$C_1 = 1 - \frac{3}{5} = \frac{2}{5}$$

Now put $y(0) = 2$

$$2 = e^0\{C_1 \cos 3(0) + C_2 \sin 3(0)$$
$$+ \frac{1}{5}(\sin 3(0) + 3\cos(0)\}$$

$2 = 1(1+0) + \frac{1}{5}(0+3)$

$2 = C_1 + 3/5$

$C_1 = \frac{7}{5}$

Now for $C_1 = 2/5$

$y_c = e^{2x}\{C_1 \cos 3x + C_2 \sin 3x\}$

~~put $C_1 = 1/5$~~

$y' = 2e^{2x}\{-3C_1 \sin 3x + 3C_2 \cos 3x\}$

put $C_1 = 2/5$

$y' = 2e^{2(0)}\{-3\,{}^{0}/_{5} \sin 3(0) + 3C_2(1)\}$

$1 = 2\{3C_2\}$

$C_2 = 1/6$

Question :- 04

Solve,

$(D^2-DD')z = \cos x \cos 2y$

Given that :-

$$(D^2-DD')z = \cos x \cos 2y$$

Required :-

we have to solve this.

Solution:-

$$(D^2-DD')z = \cos x \cos 2y$$

the given PDE can be rewrite as
$D(D-D')z = \cos x \cdot \cos 2y$ in Cf is
given by.

$$Cf = \phi_1(y) + \phi_2(y+x)$$

where its PI is given by,

$$PI = \frac{1}{(D^2-DD')} \cdot \frac{1}{2}\left[\cos(x+2y) + \cos(x-2y)\right]$$

$$= \frac{1}{2} \left[\cos(x+2y) - \frac{1}{6}\cos(x-2y)\right]$$

Hence the complete solution of PDE is,

$$z = \phi_1(y) + \phi_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y).$$