Name: Aftab Khan   ID:12985
Subject: Differential Equation

Answer 1 (a): In mathematics, a differential equation is an equation that relates one or more functions and their derivatives. In applications, the functions generally represent physical quantities, the derivatives represent their rates of change, and the differential equation defines a relationship between the two.

Example 1: dy**dx** − y**x** = 1

Example 2: dy**dx** − 3y**x** = x

Answer 1 (b): A differential equation is called **separable** if it can be written as. f(y)dy = g(x)dx. To solve a **separable differential equation**. Get all the y's on the left hand side of the equation and all of the x's on the right hand side.

Answer 1 (i):

$$y' = \frac{xy^3}{\sqrt{1+x^2}} \qquad y(0) = -1$$

First separate and then integrate both sides.

$$y^{-3}dy = x\left(1+x^2\right)^{-\frac{1}{2}}dx$$
$$\int y^{-3}dy = \int x\left(1+x^2\right)^{-\frac{1}{2}}dx$$
$$-\frac{1}{2y^2} = \sqrt{1+x^2} + c$$

Apply the initial condition to get the value of c.

$$-\frac{1}{2} = \sqrt{1} + c \qquad c = -\frac{3}{2}$$

The implicit solution is then,

$$-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Now let's solve for $y(x)$.

$$\begin{aligned}\frac{1}{y^2} &= 3 - 2\sqrt{1+x^2} \\ y^2 &= \frac{1}{3 - 2\sqrt{1+x^2}} \\ y(x) &= \pm\frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}\end{aligned}$$

Reapplying the initial condition shows us that the "–" is the correct sign. The explicit solution is then,

$$y(x) = -\frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

Let's get the interval of validity. That's easier than it might look for this problem. First, since $1 + x^2 \geq 0$ the "inner" root will not be a problem. Therefore, all we need to worry about is division by zero and negatives under the "outer" root. We can take care of both by requiring

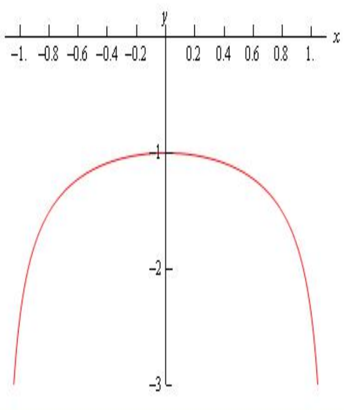
$$\begin{aligned}3 - 2\sqrt{1+x^2} &> 0 \\ 3 &> 2\sqrt{1+x^2} \\ 9 &> 4\left(1+x^2\right) \\ \frac{9}{4} &> 1 + x^2 \\ \frac{5}{4} &> x^2\end{aligned}$$

Note that we were able to square both sides of the inequality because both sides of the inequality are guaranteed to be positive in this case. Finally solving for x we see that the only possible range of x's that will not give division by zero or square roots of negative numbers will be,

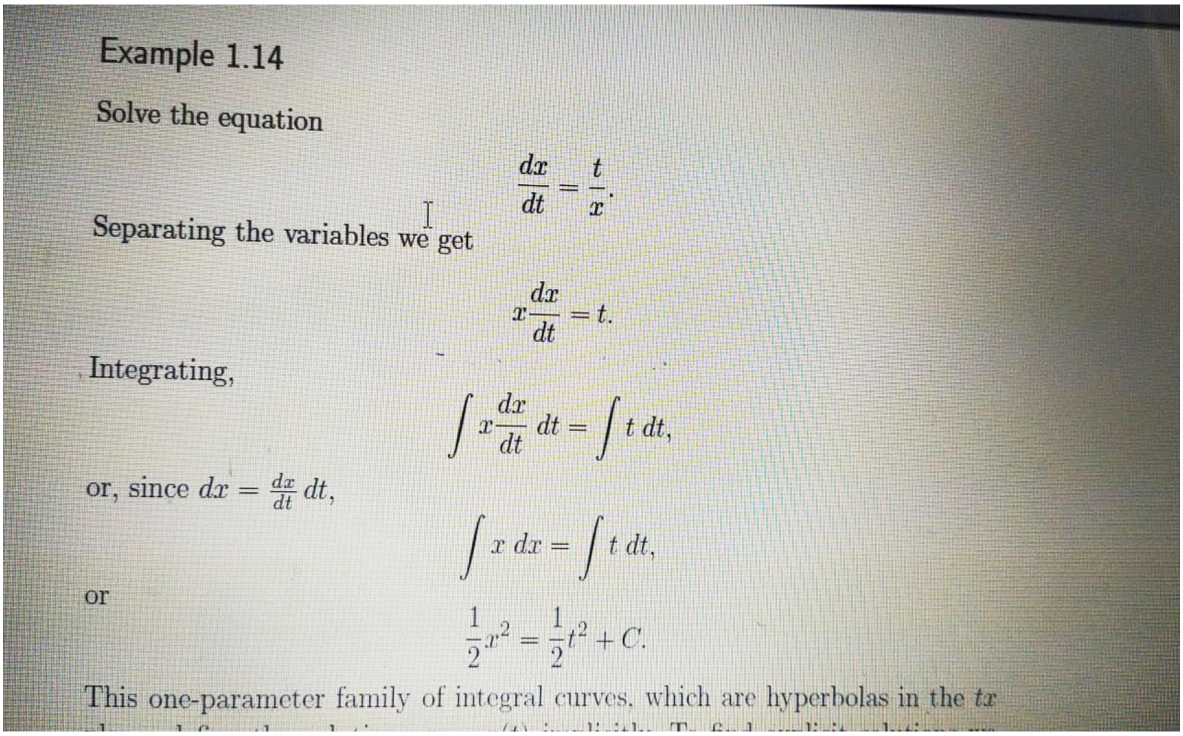
$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

and nicely enough this also contains the initial condition $x = 0$. This interval is therefore our interval of validity.

Here is a graph of the solution.



Answer 1 (ii):



Example 1.14

Solve the equation

$$\frac{dx}{dt} = \frac{t}{x}.$$

Separating the variables we get

$$x\frac{dx}{dt} = t.$$

Integrating,

$$\int x\frac{dx}{dt}\,dt = \int t\,dt,$$

or, since $dx = \frac{dx}{dt}\,dt$,

$$\int x\,dx = \int t\,dt,$$

or

$$\frac{1}{2}x^2 = \frac{1}{2}t^2 + C.$$

This one-parameter family of integral curves, which are hyperbolas in the tx

Answer 2 (a): Here we will look at solving a special class of Differential Equations called **First Order Linear Differential Equations**

First Order: They are "First Order" when there is only **dydx** , not **d²ydx²** or **d³ydx³** etc

Linear: A **first order differential equation** is **linear** when it can be made to look like this:

dy**dx** + P(x)y = Q(x)

Where **P(x)** and **Q(x)** are functions of x.

To solve it there is a special method:

- We invent two new functions of x, call them **u** and **v**, and say that **y=uv**.
- We then solve to find **u**, and then find **v**, and tidy up and we are done!

And we also use the derivative of **y=uv** (see Derivative Rules (Product Rule)):

$$dy\mathbf{dx} = udv\mathbf{dx} + vdu\mathbf{dx}$$

Steps:

Here is a step-by-step method for solving them:

- 1. Substitute **y = uv**, and

  $$dy\mathbf{dx} = udv\mathbf{dx} + vdu\mathbf{dx}$$

- into

  $$dy\mathbf{dx} + P(x)y = Q(x)$$

-

- 2. Factor the parts involving **v**
- 3. Put the **v** term equal to zero (this gives a differential equation in **u** and **x** which can be solved in the next step)
- 4. Solve using separation of variables to find **u**
- 5. Substitute **u** back into the equation we got at step 2
- 6. Solve that to find **v**
- 7. Finally, substitute **u** and **v** into **y = uv** to get our solution!

Answer 2 (i):

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1 \quad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}, \qquad 0 \le x < \frac{\pi}{2}$$

Hide Solution ▾

Rewrite the differential equation to get the coefficient of the derivative a one.

$$\begin{aligned} y' + \frac{\sin(x)}{\cos(x)}y &= 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)} \\ y' + \tan(x)y &= 2\cos^2(x)\sin(x) - \sec(x) \end{aligned}$$

Now find the integrating factor.

$$\mu(t) = \mathbf{e}^{\int \tan(x)\,dx} = \mathbf{e}^{\ln|\sec(x)|} = \mathbf{e}^{\ln\ \sec(x)} = \sec(x)$$

Can you do the integral? If not rewrite tangent back into sines and cosines and then use a simple substitution. Note that we could drop the absolute value bars on the secant because of the limits on x. In fact, this is the reason for the limits on x. Note as well that there are two forms of the answer to this integral. They are equivalent as shown below. Which you use is really a matter of preference.

$$\int \tan(x)\,dx = -\ln|\cos(x)| = \ln|\cos(x)|^{-1} = \ln|\sec(x)|$$

Also note that we made use of the following fact.

$$\mathbf{e}^{\ln f(x)} = f(x) \tag{11}$$

This is an important fact that you should always remember for these problems. We will want to simplify the integrating factor as much as possible in all cases and this fact will help with that simplification.

Now back to the example. Multiply the integrating factor through the differential equation and verify the left side is a product rule. Note as well that we multiply the integrating factor through the rewritten differential equation and NOT the original differential equation. Make sure that you do this. If you multiply the integrating factor through the original differential equation you will get the wrong solution!

$$\begin{aligned} \sec(x)y' + \sec(x)\tan(x)y &= 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x) \\ (\sec(x)y)' &= 2\cos(x)\sin(x) - \sec^2(x) \end{aligned}$$

Integrate both sides.

$$\begin{aligned} \int (\sec(x)y(x))'\,dx &= \int 2\cos(x)\sin(x) - \sec^2(x)\,dx \\ \sec(x)y(x) &= \int \sin(2x) - \sec^2(x)\,dx \\ \sec(x)y(x) &= -\frac{1}{2}\cos(2x) - \tan(x) + c \end{aligned}$$

Note the use of the trig formula $\sin(2\theta) = 2\sin\theta\cos\theta$ that made the integral easier. Next, solve for the solution.

$$\begin{aligned} y(x) &= -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + c\cos(x) \\ &= -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + c\cos(x) \end{aligned}$$

Finally, apply the initial condition to find the value of c.

$$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + c\cos\left(\frac{\pi}{4}\right)$$

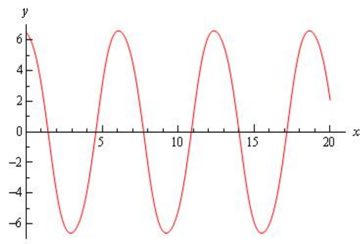
Finally, apply the initial condition to find the value of c.

$$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + c\cos\left(\frac{\pi}{4}\right)$$
$$3\sqrt{2} = -\frac{\sqrt{2}}{2} + c\frac{\sqrt{2}}{2}$$
$$c = 7$$

The solution is then.

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$$

Below is a plot of the solution.



Answer 2 (ii):

Consider the differential equation

$$x' + 2x = \sin t.$$

We multiply the DE by the integrating factor

$$\mu(t) = e^{\int 2dt} = e^{2t}$$

to get

$$(xe^{2t})' = e^{2t}\sin t.$$

Integrating both sides gives,

$$xe^{2t} = \int e^{2t}\sin t\, dt + C,$$

or

$$x(t) = e^{-2t}\int e^{2t}\sin t\, dt + Ce^{-2t}.$$

The integral on the right side can be calculated using integration by parts (or consulting an integral table). In any case we obtain the general solution

$$x(t) = e^{-2t}\left[e^{2t}\left(\frac{2}{5}\sin t - \frac{1}{5}\cos t\right)\right] + Ce^{-2t}$$
$$= \frac{2}{5}\sin t - \frac{1}{5}\cos t + Ce^{-2t}. \quad \square$$

Answer 3 (i):

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \quad y(0) = -3$$

First identify M and N and check that the differential equation is exact.

$$\begin{aligned} M &= 2xy - 9x^2 \quad M_y = 2x \\ N &= 2y + x^2 + 1 \quad N_x = 2x \end{aligned}$$

So, the differential equation is exact according to the test. However, we already knew that as we have given you $\Psi(x, y)$. It's not a bad thing to verify it however and to run through the test at least once however.

Now, how do we actually find $\Psi(x, y)$? Well recall that

$$\begin{aligned} \Psi_x &= M \\ \Psi_y &= N \end{aligned}$$

We can use either of these to get a start on finding $\Psi(x, y)$ by integrating as follows.

$$\Psi = \int M\,dx \quad \text{OR} \quad \Psi = \int N\,dy$$

However, we will need to be careful as this won't give us the exact function that we need. Often it doesn't matter which one you choose to work with while in other problems one will be significantly easier than the other. In this case it doesn't matter which one we use as either will be just as easy.

So, we'll use the first one.

$$\Psi(x, y) = \int 2xy - 9x^2\,dx = x^2y - 3x^3 + h(y)$$

Note that in this case the "constant" of integration is not really a constant at all, but instead it will be a function of the remaining variable(s), y in this case.

Recall that in integration we are asking what function we differentiated to get the function we are integrating. Since we are working with two variables here and talking about partial differentiation with respect to x, this means that any term that contained only constants or y's would have differentiated away to zero, therefore we need to acknowledge that fact by adding on a function of y instead of the standard c.

Okay, we've got most of $\Psi(x, y)$ we just need to determine $h(y)$ and we'll be done. This is actually easy to do. We used $\Psi_x = M$ to find most of $\Psi(x, y)$ so we'll use $\Psi_y = N$ to find $h(y)$. Differentiate our $\Psi(x, y)$ with respect to y and set this equal to N (since they must be equal after all). Don't forget to "differentiate" $h(y)$! Doing this gives,

$$\Psi_y = x^2 + h'(y) = 2y + x^2 + 1 = N$$

From this we can see that

$$h'(y) = 2y + 1$$

Note that at this stage $h(y)$ must be only a function of y and so if there are any x's in the equation at this stage we have made a mistake somewhere and it's time to go look for it.

We can now find $h(y)$ by integrating.

$$h(y) = \int 2y + 1\,dy = y^2 + y + k$$

You'll note that we included the constant of integration, k, here. It will turn out however that this will end up getting absorbed into another constant so we can drop it in general.

So, we can now write down $\Psi(x,y)$.

$$\Psi(x,y) = x^2y - 3x^3 + y^2 + y + k = y^2 + \left(x^2+1\right)y - 3x^3 + k$$

With the exception of the k this is identical to the function that we used in the first example. We can now go straight to the implicit solution using (4).

$$y^2 + \left(x^2+1\right)y - 3x^3 + k = c$$

We'll now take care of the k. Since both k and c are unknown constants all we need to do is subtract one from both sides and combine and we still have an unknown constant.

$$y^2 + \left(x^2+1\right)y - 3x^3 = c - k$$

$$y^2 + \left(x^2+1\right)y - 3x^3 = c$$

Therefore, we'll not include the k in anymore problems.

This is where we left off in the first example. Let's now apply the initial condition to find c.

$$(-3)^2 + (0+1)(-3) - 3(0)^3 = c \quad \Rightarrow \quad c = 6$$

The implicit solution is then.

$$y^2 + \left(x^2+1\right)y - 3x^3 - 6 = 0$$

Now, as we saw in the separable differential equation section, this is quadratic in y and so we can solve for $y(x)$ by using the quadratic formula.

$$y(x) = \frac{-\left(x^2+1\right) \pm \sqrt{\left(x^2+1\right)^2 - 4(1)\left(-3x^3-6\right)}}{2(1)}$$

$$= \frac{-\left(x^2+1\right) \pm \sqrt{x^4+12x^3+2x^2+25}}{2}$$

Now, reapply the initial condition to figure out which of the two signs in the $\pm$ that we need.

$$-3 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = -3,\ 2$$

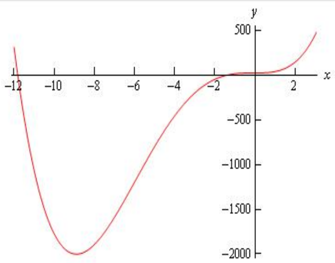
So, it looks like the "-" is the one that we need. The explicit solution is then.

$$y(x) = \frac{-\left(x^2+1\right) - \sqrt{x^4+12x^3+2x^2+25}}{2}$$

Now, for the interval of validity. It looks like we might well have problems with square roots of negative numbers. So, we need to solve

$$x^4 + 12x^3 + 2x^2 + 25 = 0$$

Upon solving this equation is zero at $x = -11.81557624$ and $x = -1.396911133$. Note that you'll need to use some form of computational aid in solving this equation. Here is a graph of the polynomial under the radical.



So, it looks like there are two intervals where the polynomial will be positive.

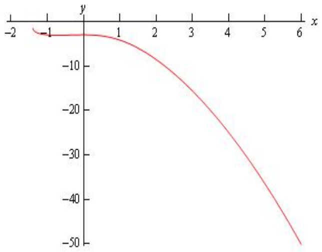
$$-\infty < x \le -11.81557624$$
$$-1.396911133 \le x < \infty$$

However, recall that intervals of validity need to be continuous intervals and contain the value of x that is used in the initial condition. Therefore, the interval of validity must be.

$$-1.396911133 \le x < \infty$$

Here is a quick graph of the solution.

Here is a quick graph of the solution.



Answer 3 (ii):

$$\frac{2ty}{t^2+1} - 2t - \left(2 - \ln\left(t^2+1\right)\right) y' = 0 \qquad y(5) = 0$$

So, first deal with that minus sign separating the two terms.

$$\frac{2ty}{t^2+1} - 2t + \left(\ln\left(t^2+1\right) - 2\right) y' = 0$$

Now, find M and N and check that it's exact.

$$M = \frac{2ty}{t^2+1} - 2t \qquad M_y = \frac{2t}{t^2+1}$$
$$N = \ln\left(t^2+1\right) - 2 \qquad N_t = \frac{2t}{t^2+1}$$

So, it's exact. We'll integrate the first one in this case.

$$\Psi(t,y) = \int \frac{2ty}{t^2+1} - 2t\,dt = y\ln\left(t^2+1\right) - t^2 + h(y)$$

Differentiate with respect to y and compare to N.

$$\Psi_y = \ln\left(t^2+1\right) + h'(y) = \ln\left(t^2+1\right) - 2 = N$$

So, it looks like we've got.

$$h'(y) = -2 \qquad \Rightarrow \qquad h(y) = -2y$$

This gives us

$$\Psi(t,y) = y\ln\left(t^2+1\right) - t^2 - 2y$$

The implicit solution is then,

$$y\ln\left(t^2+1\right) - t^2 - 2y = c$$

Applying the initial condition gives,

$$-25 = c$$

The implicit solution is now,

$$y\left(\ln\left(t^2+1\right) - 2\right) - t^2 = -25$$

This solution is much easier to solve than the previous ones. No quadratic formula is needed this time, all we need to do is solve for y. Here's what we get for an explicit solution.

$$y(t) = \frac{t^2 - 25}{\ln\left(t^2+1\right) - 2}$$

Alright, let's get the interval of validity. The term in the logarithm is always positive so we don't need to worry about negative numbers in that. We do need to worry about division by zero however. We will need to avoid the following point(s).

$$\ln\left(t^2+1\right) - 2 = 0$$

$$\begin{aligned}\ln\left(t^2+1\right)-2&=0\\ \ln\left(t^2+1\right)&=2\\ t^2+1&=\mathbf{e}^2\\ t&=\pm\sqrt{\mathbf{e}^2-1}\end{aligned}$$

We now have three possible intervals of validity.

$$\begin{gathered}-\infty<t<-\sqrt{\mathbf{e}^2-1}\\ -\sqrt{\mathbf{e}^2-1}<t<\sqrt{\mathbf{e}^2-1}\\ \sqrt{\mathbf{e}^2-1}<t<\infty\end{gathered}$$

The last one contains $t=5$ and so is the interval of validity for this problem is $\sqrt{\mathbf{e}^2-1}<t<\infty$. Here's a graph of the solution.