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Semester :- 12^{th}

Subject :- Diffjerential Equations

Section :- B

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Final Term

Qno1

(i) $W = \sin(x+ct) + \cos(2x+2ct)$.

Given::

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \quad \text{—— ①}$$

Now:-

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$= \frac{\partial}{\partial t}\left(\sin(x+ct)\right) + \frac{\partial}{\partial t}\left(\cos(2x+2ct)\right)$$

$$\frac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct)$$

Now

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[c\cos(x+ct) - 2c\sin(2x+2ct)\right]$$

$$\boxed{\frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x+ct) - 4c^2 \cos(2x+2ct)}$$

Now

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - 2\sin(2x+2ct)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct) - 2\sin(2x+2ct)\right]$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$-c^2 \sin(x+ct) - 4c^2 \cos(2x+2ct) = c^2\left[-\sin(x+ct - 4\cos(2x+2ct)\right]$$

$$-c^2 \sin(x+ct) - 4c^2 \cos(2x+2ct) = -c^2 \sin(x+ct) - 4c^2\cos(2x+2ct)]$$

$0 = 0$ (satisfied)

1 (ii)

$w = \tan(2x + ct)$

Now

$$\frac{\partial w}{\partial t} = c\,Sec^2(2x + ct)$$

& $$\frac{\partial^2 w}{\partial t} = \frac{\partial}{\partial t}\left(c\,Sec^2(2x + ct)\right)$$

$$= c^2 \cdot 2\,Sec^2(2x + ct)\tan(2x + ct)$$

Now

$$\frac{\partial w}{\partial x} = 2\,Sec^2(2x + ct)$$

$$\frac{\partial^2 w}{\partial x^2} = 4\,Sec^2(2x + ct)\tan(2x + ct)$$

$$4c^2 \sec^2(2x+ct)\tan(2x+ct) = 4c^2 \sec^2(2x+ct)\tan(2x+ct$$

$0 = 0$ (Satisfied

Qno2 :-

Expand the following function in fourier series

$F(x) = x \quad -\pi < x \leq 0$

$\quad -2x, \quad 0 \leq x \leq \pi$

Ans

Given function is

$$F(x) = \begin{cases} x ; & -\pi < x \leq 0 \\ 2x ; & 0 \leq x \leq \pi \end{cases}$$

we have to find the fourier coefficient a_0, a_n & b_n

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left[0 - \frac{\pi^{\cancel{2}}}{2}\right] + \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^{\cancel{2}}}{\cancel{2}} - 0\right]$$

$$\boxed{a_0 = -\frac{\pi}{2} + \pi = \pi/2} \quad - (1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(x \cos nx\,dx) + \frac{1}{\pi}\int_{0}^{\pi}(2x \cos nx)\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+\frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right)-\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$a_n=\frac{1}{\pi}\left[\frac{\cos(0)}{n^2}-\frac{\cos nx}{n^2}\right]+\frac{2}{\pi}\left[\frac{\cos nx}{n^2}-\frac{\cos(0)}{n^2}\right]$$

$$=\frac{1}{\pi}\left[\frac{1-(-1)^n+2(-1)^n-2}{n^2}\right]=\frac{(-1)^n-1}{\pi n^2}$$

So

$$\boxed{a_n=\begin{cases}\frac{-2}{\pi n^2}; & \text{if } n \text{ is odd}\\ 0; & \text{if } n \text{ is even}\end{cases}} \longrightarrow (2)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}F(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}x\sin nx\,dx$$

$$+\frac{2}{\pi}\int_{0}^{\pi}x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$\boxed{b_n = \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] = -\frac{3\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}}$$

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So the required fourier Series ~~no~~ is

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2x-1)x}{(2x-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

Qno3:-

Solve the initial value problem

$y'' - 4y' + 13y = 8\sin 3x$, $y(0) = 1$ and

$y'(0) = 2$

Solution:-

Given

$y'' - 4y' + 13y = 8\sin 3x$

we have to find $y = y_c + y_p$

For y_c the characteristic (auxilary Eqn)

is :

$m^2 - 4m + 13 = 0$

$\Rightarrow m = \dfrac{4 \pm \sqrt{16-52}}{2} \Rightarrow m = \dfrac{4 \pm 6i}{2}$

$\Rightarrow m = 2 \pm 3i$; α 2 & $\beta = 3$

So $y_c = e^{2x}\{C_1 \cos 3x + C_2 \sin 3x\}$

For y_p Let

$y_p = \text{Imag}\left(\dfrac{1}{m^2 - 4m + 13} \; 8e^{3ix}\right)$

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$$8\,\mathrm{Imag}\ \frac{e^{3jx}}{(3j)^2-4(3i)+13}$$

$$= 8\,\mathrm{Imag}\ \frac{e^{3ix}}{-9-12j+13}$$

$$8\,\mathrm{Imag}\cdot\frac{e^{3ix}}{4-12i}$$

$$Yp = 2\,\mathrm{Imag}\cdot\frac{e^{3jx}}{(1-3i)}\times\frac{(1+3i)}{(1+3i)}$$

$$Yp = 2\,\mathrm{Imag}\cdot\frac{(1+3i)(e^{3ix})}{(1)^2\ (3i)^2}$$

$$Yp = 2\,\mathrm{Imag}\ \frac{(1+3j)(e^{3ix})}{10}$$

$$Yp = \frac{2}{10}\left(\mathrm{Imag}\,(1+3j)(\cos 3x + j\sin 3x)\right)$$

$$Yp\ \frac{2}{10}\left(\sin 3x + 3\cos 3x\right)$$

So the general solution is

$$y = y_c + y_p$$

$$y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x + \frac{2}{10}\left[\sin 3x + 3\cos 3x\right]$$

Now use the initial condition $y(0)=1$

$$y(0) = C_1 e^{(0)} \cos(0) + C_2 e^{(0)} \sin(0) + \frac{2}{10}\left[\sin(0) + 3\cos(0)\right]$$

$$1 = C_1(1) + 0 + 0 + \frac{2}{10}(3(1))$$

$$1 = C_1 + \frac{6}{10} \Rightarrow$$

$$C_1 = 1 - \frac{6}{10} = \frac{4}{10} = \frac{2}{5}$$

Again use another initial condition

$$y'(0) = 2$$

So

$$y' = C_1 2e^{2x} \cos 3x + C_1 e^{2x}(-3\sin 3x)$$
$$+ C_2 2e^{2x} \sin 3x + C_2 e^{2x}(3\cos 3x)$$
$$+ \frac{2}{10}(\cos 3x - 3\sin 3x)$$

$$y'(0) = C_1 2e^{(0)} \cos(0) + C_1 e^{(0)}(-3\sin(0))$$
$$+ C_2 2e^{(0)} \sin(0) + C_2 e^{(0)}(3\cos(0))$$
$$+ \frac{2}{10}(\cos(0) - 3(\sin(0)))$$

$$2 = 2C_1 + 0 + 0 + C_2 3(1) + \frac{2}{10}(1 - 3(0))$$
$$2 = 2C_1 + 3C_2 + \frac{2}{10}$$

$$2 = 2(2/5) + 3C_2 + 2/10 \qquad \boxed{\text{use } C_1 = 2/5}$$

$$\frac{1}{3}\left(2 - \frac{4}{5} - \frac{2}{10}\right) = C_2$$

$$\boxed{C_2 = \frac{1}{3}\left(\frac{20-8-2}{10}\right) = 1/3}$$

So the general Solution is

$$Y = \frac{2}{5} e^{2x} \cos 3x + \frac{1}{3} e^{2x} \sin 3x + \frac{2}{10}\left[\sin 3x + 3\cos 3x\right]$$

~~the required Solution~~

Qno 4

Solve :-

$(D^2 - DD')z = \cos x \cos 2y$

⇒ Solution :-

It is already in symbolic form.

$(D^2 - DD')z = \cos x \cos 2y$ — ⓐ

Put A.E $D^2 - DD' = 0$

As we know

$\frac{D}{D'} = m$ ie $D = m$, $D' = 1$

⇒ $m^2 - m = 0$

$m = 0, 1$

There fore $C.F = f_1(y) + f_2(y + x)$

from Eq ⓐ

$$P.I = \frac{1}{D^2 - DD'} \cos x \cos 2y$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 - DD'} 2 \cos x \cos 2y$$

As $2 \cos A \cos B = \cos(A+B + \cos(A-B)$

$Cf = f_1(y-x) + x f_2(y-x)$

$$PI = \frac{1}{D^2 + 2DD' + D'^2} \left[2(y-x) + \sin(x-y)\right]$$

$$= \frac{1}{(D+D')^2} \left[2(y-x) + \sin(x-y)\right]$$

By General method

$m = -1 ; y - x = c$

$$= \frac{1}{D+D'} \int \left[2c + \sin(-c)\right] dx$$

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$$= \frac{1}{D+D'}\left[2\,(x - (\sin c)\,x\right]$$

Replaciny by $y-x$

$$= \frac{1}{D+D'}\left[2x(y-x) - x\sin(y-x)\right]$$

Again Put $y - x = c$

$$= \int (2xc - x\sin c)\,dx \Rightarrow (x^2 - \frac{x^2}{2}\sin c$$

Replacing by $y-x$

$$= x^2(y-x) - \frac{x^2}{2}\sin(y-x) = x^2y - x^3 +$$

$$\frac{x^2}{2}\sin(x-y)$$

Hence the required solution is

$$z = C.F + P.I = f_1(y-x) + f_2(y-x) + x^2y - x^3$$
$$+ \tfrac{1}{2}x^2\sin(x-y)$$