

(1) Application of Partial differential equation:

Many engineering problems are governed by different types of partial differential equations, and some of the more important types are given below.

Tricomi equation:-

$$y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \begin{cases} y > 0 : \text{elliptic} \\ y < 0 : \text{hyperbolic} \end{cases}$$

Laplace equation (or variants): $\partial^2 \phi / \partial x^2 + \partial^2 \phi / \partial y^2 = 0$
 $\nabla^2 \phi = 0$

Poisson's equation:-

$$\partial^2 \phi / \partial x^2 + \partial^2 \phi / \partial y^2 = f(x, y)$$

Helmholtz equation:

$$\partial^2 \phi / \partial x^2 + \partial^2 \phi / \partial y^2 + c^2 \phi = 0$$

Plate bending:-

$$\nabla^2 \nabla^2 w = \nabla^4 w = qD$$

(2)

Wave equation:-

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

Fourier equation:-

$$\partial T \partial t = a (\partial^2 T \partial x^2)$$

Separable differential equations:-

For equations which can be expressed in separable form as shown below, the solution can be obtained easily as,

$$\frac{dy}{dx} = F(x, y) \quad \frac{dy}{g(y)} = f(x) dx \quad \int \frac{dy}{g(y)} =$$

$$\int f(x) dx + C$$

$$M(x, y) dx + N(x, y) dy = 0 \quad M(x) dx = N(y) dy$$

$$\text{then } \int M(x) dx = - \int N(y) dy + C.$$

3)

Separable differential equations:-

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$$\frac{dy}{dx} = F(x, y) \quad \frac{dy}{g(y)} = f(x) dx \quad \int \frac{dy}{g(y)} = \int f(x) dx + C$$

$$(x) dx + C$$

$$M(x, y) dx + N(x, y) dy = 0 \quad M(x) dx = -N(y) dy$$

$$\text{then } \int M(x) dx = - \int N(y) dy + C$$

Example:-

$$\frac{dy}{dx} = x^3 + (y^2 + 1) \Rightarrow \frac{dy}{y^2 + 1} = x^3 dx$$

$$\int \frac{dy}{y^2 + 1} = \int x^3 dx + C \Rightarrow \tan^{-1} y = \frac{1}{4} x^4 + C$$

$$\Rightarrow y = \tan \left(\frac{1}{4} x^4 + C \right)$$

Example:-

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \quad \text{subject to } y(0) = -1$$

4)
Since is a separable function, the problem can be solved as.

$$2(y-1) dy = (3x^2 + 4x + 2) dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

Example:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \quad \text{subject to } y(0) = -1$$

Since this is a separable function, the problem can be solved as

$$2(y-1) dy = (3x^2 + 4x + 2) dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

Based on the boundary condition, $c = 3$,

$$\text{hence } y^2 - 2y = x^3 + 2x^2 + 2x + 3.$$

This quadratic equation in y^2 can be solved with two solutions by the quadratic equation

as:

$$y = 1 \quad \sqrt{x^3 + 2x^2 + 2x + 4} \quad \text{and} \quad y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4}$$

Since the second solution does not satisfy the boundary condition, it will not be accepted; hence, the solution to this differential equation is obtained.

Variation of Parameters:-

for the following equation form, it is possible to solve it by variation of parameters.

$$\text{for } \frac{dy}{dx} = P(x)y + Q(x)$$

Put $y = C(x) e^{\int P(x) dx}$ by differentiating,

it gives.

$$\frac{dy}{dx} = \frac{dC(x)}{dx} e^{\int P(x) dx} + \underbrace{C(x) P(x) e^{\int P(x) dx}}_{P(x)y}$$

Substitute it to the original ODE

6)

$$\frac{dc(x)}{dx} = Q(x) e^{-\int P(x) dx} \quad \text{Comparing the}$$

terms, it given

$$C(x) = \int Q(x) e^{-\int P(x) dx} dx + \bar{C}$$

Example:-

$$(x+1) \frac{dy}{dx} - ny = e^x (x+1)^{n+1}$$

This equation is now expressed as,

$$\frac{dy}{dx} = P(x) y + Q(x)$$

$$\frac{dy}{dx} = \frac{n}{x+1} y + \frac{e^x (x+1)^n}{Q(x)}$$

for $x \neq -1$

Solving the homogeneous part of the ODE

$$\frac{dy}{dx} = \frac{n}{x+1} y \quad \text{then} \quad \frac{dy}{y} = \frac{n}{x+1} dx$$

$$\ln |y| = \ln |x+1|^n + c_1$$

$$y = c (x+1)^n$$

7)

Look for solution $y = c(x) (x+1)^n$

where $c(x)$ is the variation of parameters.

Substitute it to the ODE.

$$\frac{dc(x)}{dx} (x+1)^n + nc(x) (x+1)^{n-1} = nc(x)$$

$$(x+1)^{n-1} = nc(x) (x+1)^{n-1} + e^x (x+1)^n$$

$$\frac{dy}{dx} = \frac{n}{n+1} y + e^x (x+1)^n$$

Comparison gives $\frac{dc(x)}{dx} = e^x$

Integration of this equation gives.

$$c(x) = e^x + \bar{C}$$

General solution is hence given by

$$y = (x+1)^n (e^x + \bar{C})$$

8)
The Bernoulli equation is an important equation type which can be solved in a similar way by variation of parameters. Consider the following form of equation.

$$\frac{dy}{dx} = P(x)y + Q(x)y^n$$

step 1:- put $z = y^{1-n}$

step 2:- Then $\frac{dz}{dx} = (1-n)y^{-n} \frac{dy}{dx}$

$$\frac{dz}{dx} = (1-n)P(x)z + (1-n)Q(x)$$

The non linear ODE now becomes linear ODE. It can be solved by formula.

step 3:- $n = -1, z = y^2$. Inverting z to get y

$$\frac{dy}{dx} = \frac{y}{2x} + \frac{x^2}{2y}$$

$$\frac{dz}{dx} = \frac{1}{x}z + x^2$$

9)

$$z = e^{\int \frac{1}{x} dx} \left(\int x^2 e^{-\int \frac{1}{x} dx} dx + c \right) = Cx + \frac{1}{2} x^3$$

Back substitution of $z = y^2$ of $z = y^2$

$$y^2 = Cx + \frac{1}{2} x^3$$

Homogeneous equations:-

for equation of the following type, where all the coefficients are constant, it can be evaluated according to different.

Laplace equation:-

Laplace equation forms an important governing condition for many types of problems. Some of the more common forms are given by.

Three dimensional Laplace equation:-

$$u_{xx} + u_{yy} + u_{zz} = 0$$

10)

Two dimensional heat Conduction:-

$$a^2 (u_{xx} + u_{yy}) = u_t$$

Two dimensional Seepage Problem.

$$(K_x u_{xx} + K_y u_{yy}) = 0$$

There are two major types of boundary conditions to this problem.

Dirichlet problem:-

Boundary conditions prescribed

as u .