ID #16950 Muhammad Asif

(1)

Q1:

(a) Define 2nd order linear homogenous/non-homogenous differential equation along with example?

Definition:-

(Homogeneous/Nonhomogeneous Equation)

The linear differential equation

$y'' + p(x)y' + q(x)y = f(x) \rightarrow (1)$ is homogeneous if the function f on the right side is 0 for all $x \in I$. In this case equation becomes

$$y'' + p(x)y' + q(x)y = 0 \cdot \Rightarrow (2)$$

Equation (1) is non-homogeneous if f is not zero function on I, i.e. (1) is non-homogeneous if $(x) \neq 0$ for some $x \in I$.

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(a) $4y'' - 6y' + 7y = 0$

The auxiliary equation is

$4D^2 - 6D + 7 = 0$

$a = 4, \; b = -6, \; c = 7.$

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36 - 112}}{2(4)}$$

$$= \frac{6 \pm \sqrt{-76}}{8}$$

$$= \frac{6 \pm i\sqrt{76}}{8}$$

$$= \frac{6 \pm i\sqrt{4 \times 19}}{8}$$

$$= \frac{6 \pm i\,2\sqrt{19}}{8}$$

$$= \frac{3 \pm i\sqrt{19}}{4}$$

$$D = \frac{3}{4} \pm i\frac{\sqrt{19}}{4}$$

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Since the root are complex conjugate hence its solution is

$$y = e^{3/4 x}\left\{A \cos\frac{\sqrt{19}}{4}x + B\sin\frac{\sqrt{19}}{4}x\right\}$$

(ii) $y'' - 4y' - 12y = 3e^{5x}$

Solution

To find $y_c = ?$

$y'' - 4y' - 12y = 0$

The auxiliary equation is

$D^2 - 4D - 12 = 0$

$D^2 - 6D + 2D - 12 = 0$

$D(D-6) + 2(D-6) = 0$

$(D+2)(D-6) = 0$

$D + 2 = 0, \; D - 6 = 0.$

$D = -2, \; D = 6$

Since root are real distinct Hence

$y_c = Ae^{-2x} + Be^{6x}$

Now to find y_p:

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$y_p = Ce^{5x}$

Hence the required Solution is

$$y = y_c + y_p$$

$$y = Ae^{2x} + Be^{6x} + Ce^{5x}.$$

Ques:-

(iv) $y'' - 8y' + 17y = 0 \quad y(0) = 4,\ y'(0) = -1$

Solution

The auxiliary equation is

$$D^2 - 8D + 17 = 0$$

$$a = 1,\ b = -8,\ c =$$

$$D = \frac{8 \pm \sqrt{64-68}}{2}$$

$$D = \frac{8 \pm \sqrt{-4}}{2}$$

$$= \frac{8 + i2}{2} = 4 \pm i$$

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The roots are complex conjugate

Hence

$$y = e^{4x}\{A\cos x + B\sin x\}$$

$$y(0) = 4$$

$$i(A\cos 0 + B\sin 0) = 4$$

$$A = 4$$

$$y' = 4e^{4x}\{A\cos x + B\sin x\} + e^{4x}\{-A\text{-}\sin x + B\cos x\}$$

$$y'(0) = 4e^{0}\{A + 0\} + e^{0}\{0 + B\}$$

$$y'(0) = 4A + B$$

$$-1 = 4A + B$$

$$4A + B = -1$$

$$4(4) + B = -1$$

$$16 + B = -1$$

$$B = -1 - 16$$

$$\boxed{B = -17}$$

$$y = e^{4x}\{4\cos x - 17\sin x\}.$$

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(iii) $y'' - 4y' + 9y = 0 \quad y(0) = 0\ y'(0) = -8$

Solution:-

$$y'' - 4y' + 9y = 0$$

$$y(0) = 0, \quad y'(0) = -8$$

$$D^4 - 4D + 9 = 0$$

$$a = 1, b = -4, c = 9$$

$$D = \frac{4 \pm \sqrt{16-36}}{2}$$

$$= \frac{4 \pm \sqrt{-20}}{2}$$

$$= \frac{4 \pm i2\sqrt{2}}{2}$$

$$= 2 \pm i\sqrt{5}$$

Since the roots are complex conjugate, Here

$$y = e^{2x}\{A\cos\sqrt{5}x + B\sin\sqrt{5}x\}$$

$$y(0) = 0$$

$$e^0\{A + 0\} = 0$$

$$A = 0$$

$$y = e^{2x} B\sin\sqrt{5}x$$

$$y' = 2e^{2x} B\sin\sqrt{5}x + e^{2x}B\sqrt{5}\cos\sqrt{5}x$$

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$y'(0) = 0 + eB\sqrt{5}$

$-8 = \sqrt{5}\ B$

$B = -8/\sqrt{5}$

$y = e^{2x}\{0 - \frac{8}{\sqrt{5}} \sin\sqrt{5}x$

$y = \frac{-8}{\sqrt{5}} e^{2x} \sin\sqrt{5}x$..

(ii) $y'' + 14y' + 49y = 0$ $y(-4) = -1$ $y'(-4) = 5$

Sol:-

The auxiliary equation

$D^2 + 14D + 49 = 0$

$D^2 + 7D + 7D + 49 = 0$

$D(D+7) + 7(D+7) = 0$

$(D+7)(D+7) = 0$

$D = -7, -7$

Since the roots are real and repeated Hence

$y = (A+Bx)e^{-7x}$

$y(-4) = -1$

$\{A + B(-4)\} e^{-7(-4)} = -1$

$(A - 4B) e^{-28} = -1$

$A - 4B = -1/e^{-28}$

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$$A - 4B = -e^{28} \quad \text{—(i)}$$

$$y = (A + Bx)e^{-7x}$$

$$y' = -7e^{-7x}\{A + Bx\} + e^{-7x}\{B\}$$

$$= -7e^{-7x}\{A + Bx\} + Be^{-7x}$$

$$y'(-4) = -7e^{-28}\{A - 4B\} + Be^{-28}$$

$$5 = e^{-28}\{-7A + 28B + B\}$$

$$\frac{5}{e^{-28}} = -7A + 29B$$

$$-7A + 29B = 5e^{28} \quad \text{—> (ii)}$$

(x)ing (i) by 7 and add it with eq (ii)

$$7A - 28B = -7e^{+28}$$
$$-7A + 29B = 5e^{28}$$

$$B = -2e^{28}$$

eq (i) $A - 4(-2e^{28}) = -e^{28}$

$$A + 8e^{28} = -e^{28}$$

$$A = -8e^{28} - e^{28}$$

$$A = -9e^{28}$$

$$y = (-9e^{28} - 2e^{28}x)e^{-7x}$$
$$= (-9 - 2x)e^{28}e^{-7x}$$

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$= -9 - 2x e^{-7}(x-4)$ Ans.

(ii) $16y'' - 40y' + 25y = 0 \quad y(0) = 3 \quad y'(0) = -9/4$

Sol:

The auxiliary equation is

$$16D^2 - 40D + 25 = 0$$

$$16D^2 - 20D - 20D + 25 = 0$$

$$4D(4D - 5) - 5(4D - 5) = 0$$

$$(4D - 5)(4D - 5) = 0$$

$$4D - 5 = 0, \quad 4D - 5 = 0$$

$$D = 5/4, \quad D = 5/4.$$

The roots are real and repeated

Hence

$$y = e^{5/4 x}\{A + Bx\}$$

$$y(0) = e^0\{A + 0\}$$

$$3 = A$$

$$y = e^{5/4 x}\{3 + Bx\}$$

$$y' = \frac{5}{4} e^{5/4 x}\{3 + Bx\} + e^{5/4 x} B$$

$$y'(0) = \frac{5}{4} \cdot 3 + B$$

$$\frac{-9}{4} = \frac{15}{4} + B$$

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$$B = \frac{-9}{4} - \frac{15}{4}$$

$$= \frac{-9-15}{4} = \frac{-\cancel{24}\,6}{\cancel{4}\,1}$$

$$B = -6$$

$$y = \{A + Bx\}\, e^{5/4\,x}$$

$$= \{3 - 6x\}\, e^{5/4\,x}$$

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(i)

Q3 Define laplace transform along with example?

Laplace Transform:-

The laplace transform of a function $f(t)$ is defined by the integral

$$\mathcal{L}(f;s) = \int_0^\infty e^{-st} f(t)\,dt$$

for those s where the integral coverges. Here s is allowed to take complex values.

Example:-

$$\mathcal{L}(\cos(wt)).$$

Solution we use the formula.

$$\cos(wt) = \frac{e^{iwt} + e^{-iwt}}{2}$$

so

$$\mathcal{L}(\cos(wt);s) = \frac{1/(s-iwt) + 1/(s+iwt)}{2}$$

$$= \frac{s}{s^2+w^2}.$$

(ii)

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(A) Find the laplace transforms of the given function.

(i)

$$g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t).$$

$$f(t) = e^{3t} - e^{3t}\cos(6t) + \cos(6t)$$

$$y(t) = t^2 \sin(2t)$$

$$h(t) = (10t)^{3/2}$$

$$h(t) = \begin{cases} t^4 & t < 5 \\ t^4 + 3\sin\left(\frac{t}{10} - \frac{1}{2}\right) & t \geq 5 \end{cases}$$

$$w(t) = \begin{cases} t & t < 6 \\ -8 + (t-6)^2 & t \geq 6. \end{cases}$$

$$H(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^5}$$

$$f(s) = \frac{3}{s^2+25} + \frac{6s}{s^2+25}$$

$$G(s) = \frac{1-3s}{s^2-8s+21}$$

$$w(s) = \frac{86s-78}{(s+3)(s-4)(5s-1)}$$

$$Y(s) = \frac{25}{s^3(s^2+4s+5)}$$

$$H(s) = \frac{se^{-4s}}{(3s+2)(s-2)}.$$

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(ii) $f(t) = 6(e^{\wedge}-5t) + e^{\wedge}3t + 5(t^{\wedge}3) - 9$

$$F(s) = \mathcal{L}\{f(t)\} = 6\mathcal{L}\{e^{-5t}\} + \mathcal{L}\{e^{3t}\} + 5\mathcal{L}\{t^3\} - 9\mathcal{L}(1)$$

$$= 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 9\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}.$$

(iii) $h(t) = e^{\wedge}3t + \cos(6t) - e^{\wedge}(3t)\cos(6t).$

$$H(t) = \mathcal{L}\{h(t)\} = \mathcal{L}\{e^{3t}\} + \mathcal{L}\{\cos(6t)\} - \mathcal{L}\{e^{3t}\cos(6t)\}$$

$$= \frac{1}{s-3} + \frac{s}{s^2+6^2} - \frac{s-3}{(s-3)^3+6^3}$$

$$= \frac{1}{s-3} + \frac{s}{s+36} - \frac{s-3}{(s-3)^3+36}$$

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Q4

(i) $y'' - 10y' + 9y = 5t, \quad y(0) = -1, \; y'(0) = 2.$

Applying the laplace transform to bothside we find

$$(s^2 - 10s + 9)Y + s - 2 - 10 = \frac{5}{s^2} \Rightarrow Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

To find the inverse laplace transform we will need first simplify the expression for $Y(s)$ using the partial fraction decomposition.

$$\frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

we find

$$B = \frac{5}{9}, D = -2, C = \frac{31}{81}, A = \frac{50}{81}$$

Therefore, using the linearity of the inverse laplace transform

$$y(t) = \frac{50}{81} + \frac{5t}{9} + \frac{31}{81} e^{9t} - 2e^{t}.$$

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(ii) $y''-6y'+15y=2\sin(3t)$, $y(0)=-1$, $y'(0)=-4$

Ans we have

$$(s^2-6s+15)Y+s-2=\frac{6}{s^2+9} \Rightarrow Y(s)=\frac{-s^3+9s^2-9s+24}{(s^2+9)(s^2-6s+15)}$$

$$=\frac{As+B}{s^2+9}+\frac{Cs+D}{s^2-6s+15}$$

To find the constants, we need to simplify the expression on the right and equate the coefficients at ~~the~~ equal powers.

s^3: $A+C=-1$

s^2: $-6A+B+D=2$

s^1: $15A-6B+9C=-9$

s^0: $15B+9D=24$

Solution is

$$A=\frac{1}{10},\ B=\frac{1}{10},\ C=\frac{11}{10},\ D=\frac{5}{2}$$

Now, we need to find the inverse, laplace transform, let us start with the first term.

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}+\frac{1}{s^2+9}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$+\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}=\cos 3t+\frac{1}{3}\sin 3t$$

Rearrange the expression in the following way that we can always add and subtract.

(iii)

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$$\frac{-11s+25}{s^2-6s+15} = \frac{-11s+25}{(s-3)^2+6}$$

$$= \frac{-11(s-3)-8}{(s-3)^2+6}$$

$$= -11\frac{(s-3)}{(s-3)^2+6} - \frac{8}{\sqrt{6}}\frac{\sqrt{6}}{(s-3)^2+6}$$

Now

$$\mathcal{L}^{-1}\left\{\frac{-11s+25}{s^2-6s+15}\right\} = -11e^{3t}\cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t}\sin\sqrt{6}t.$$

The final answer is

$$y(t) = \mathcal{L}^{-1}\{Y\} = \frac{1}{10}\left(\cos 3t + \frac{1}{3}\sin 3t - 11e^{3t}\cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t}\sin\sqrt{6}t\right).$$