"FINAL TERM PAPER"

ID # 16236.
BS : SE (SEC'A').
Course : Linear Algebra.
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Q#1. Determine if the following System is consistent or not.

$$x_1 - (3^{rd} - ID)\,x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$5x_1 - 5x_3 = 10.$$

Sol :-

$$\left[\begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ 5x_1 - 5x_3 = 10 \end{array}\right]$$

Matrix with the co-efficients and solutions.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

Reduce matrix to row echelon form

Swap matrix rows : $R_1 \leftrightarrow R_3$

$$= \begin{bmatrix} 5 & 0 & -5 & 10 \\ 0 & 2 & -8 & 8 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$

$R_3 \leftarrow R_3 - 1/5 \cdot R_1$

$$\sim \begin{bmatrix} 5 & 0 & -5 & 10 \\ 0 & 2 & -8 & 8 \\ 0 & -2 & 2 & -2 \end{bmatrix}$$

$R_3 \leftarrow R_3 + 1 \cdot R_2$

$$\sim \begin{bmatrix} 5 & 0 & -5 & 10 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & -6 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & 0 & -5 & 10 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & -6 & 6 \end{bmatrix}$$

Multiply matrix row by constant

$R_3 \leftarrow -1/6 \cdot R_3$

$$\sim \begin{bmatrix} 5 & 0 & -5 & 10 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$R_2 \leftarrow R_2 + 8 \cdot R_3$

$$\sim \left|\begin{matrix} 5 & 0 & -5 & 10 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{matrix}\right|$$

$R_1 \leftarrow R_1 + 5 \cdot R_3$

$$\sim \left|\begin{matrix} 5 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{matrix}\right|$$

$R_2 \leftarrow \frac{1}{2} \cdot R_2$

$$\sim \left|\begin{matrix} 5 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{matrix}\right|$$

$R_1 \leftarrow \frac{1}{5} \cdot R_1$

$$\sim \left|\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{matrix}\right|$$

The Solutions to the System of equations are, $X1 = 1$, $X2 = 0$, $X3 = -1$.

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Ans.

$$\text{Inverse} = \frac{1}{D} \cdot \text{Adj}(\ ).$$

$$\therefore D = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 3 \\ 5 & -2 & 7 \end{vmatrix}$$

$$\Rightarrow D = 6$$

$$, \text{Adj} = \begin{bmatrix} -1 & -38 & 17 \\ 1 & -4 & 1 \\ 1 & 26 & -11 \end{bmatrix}$$

$$\therefore \text{Inverse} = \frac{1}{6} \begin{bmatrix} -1 & -38 & 17 \\ 1 & -4 & 1 \\ 1 & 26 & -11 \end{bmatrix}$$

Result

Q#02:- Linear System by Guass-Jordan Method;

$$2x + 2y + 4z = 18$$
$$x + 3y + 2z = 13$$
$$3x + 2y - 3z = 14$$

Sol:- The above given linear system can be written in matrix form as;

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 18 \\ 1 & 3 & 2 & 13 \\ 3 & 2 & -3 & 14 \end{array}\right]$$

$$\downarrow$$

$$\therefore \begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 2 & -3 \end{bmatrix}$$, solving by Guass Jordan,

I: Echelon Form;

P.T.O

$$R_1 \longleftrightarrow R_3$$

$$\sim \begin{bmatrix} 3 & 2 & -3 \\ 1 & 3 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$R_2 - \tfrac{1}{3}R_1 \longrightarrow R_1$$

$$\sim \begin{bmatrix} 3 & 2 & -3 \\ 0 & 7/3 & 3 \\ 2 & 2 & 4 \end{bmatrix}$$

$$R_3 - \tfrac{2}{3}R_1 \longrightarrow R_3$$

$$\sim \begin{bmatrix} 3 & 2 & -3 \\ 0 & 7/3 & 3 \\ 0 & 2/3 & 6 \end{bmatrix}$$

$$R_3 - \tfrac{2}{7}R_2 \longrightarrow R_3$$

$$\sim \begin{bmatrix} 3 & 2 & -3 \\ 0 & 7/3 & 3 \\ 0 & 0 & 36/7 \end{bmatrix}.$$

Now, further Reduce the matrix to Reduced Row Echelon form;

$$\therefore \begin{bmatrix} 3 & 2 & -3 \\ 0 & 7/3 & 3 \\ 0 & 0 & 36/7 \end{bmatrix}$$

$$\tfrac{7}{36} \cdot R_3 \longrightarrow R_3$$

$$\sim \begin{bmatrix} 3 & 2 & -3 \\ 0 & 7/3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - 3R_3 \longrightarrow R_2$$

$$\sim \begin{bmatrix} 3 & 2 & -3 \\ 0 & 7/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 + 3 \cdot R_3 \longrightarrow R_1$$

$$\sim \begin{bmatrix} 3 & 2 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tfrac{3}{7} \cdot R_2 \longrightarrow R_2$$

$$\sim \begin{bmatrix} 3 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(P.T.O)

$R_1 - 2.R_2 \rightarrow R_1$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{1}{3}.R_1 \rightarrow R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Hence, the linear system solved by Gauss-Jordan method Gives us;

$x = 1, \quad y = 1, \quad z = 1$

= Result.

= Result

Q#05 To describe the sol. set if the Given homogeneous system has non-trivial Solution;

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$-3x_1 - 2.5x_2 + 4x_3 = 0$$
$$6x_1 + x_2 - 8x_3 = 0$$

Sol:

As Given;

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$-3x_1 - 25x_2 + 4x_3 = 0$$
$$6x_1 + x_2 - 8x_3 = 0$$

In Matrix Form;

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -25 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]$$

we will solve the solution of these equations by Gaussian Elimination;

∴ Reduce matrix to Row Echelon Form;

i.e:
$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -25 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

$$R_1 \longleftrightarrow R_3$$

$$\sim \begin{bmatrix} 6 & 1 & -8 & 0 \\ -3 & -25 & 4 & 0 \\ 3 & 5 & -4 & 0 \end{bmatrix}$$

$$R_2 + \tfrac{1}{2} \cdot R_1 \longrightarrow R_2$$

$$\sim \begin{bmatrix} 6 & 1 & -8 & 0 \\ 0 & -49/2 & 0 & 0 \\ 3 & 5 & -4 & 0 \end{bmatrix}$$

$$R_3 - \frac{1}{2} \cdot R_1 \longrightarrow R_3$$

$$\sim \begin{bmatrix} 6 & 1 & -8 & 0 \\ 0 & -49/2 & 0 & 0 \\ 0 & 9/2 & 0 & 0 \end{bmatrix}$$

$$R_3 + \frac{9}{49} \cdot R_2 \longrightarrow R_3$$

$$\sim \begin{bmatrix} 6 & 1 & -8 & 0 \\ 0 & -49/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now,

Reduce Matrix to Reduced Row Echelon Form;

$$\begin{bmatrix} 6 & 1 & -8 & 0 \\ 0 & -49/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\frac{2}{49} \cdot R_2 \longrightarrow R_2$$

$$\sim \begin{bmatrix} 6 & 1 & -8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 1 \cdot R_2 \rightarrow R_1$

$$\sim \begin{bmatrix} 6 & 0 & -8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{1}{6} \cdot R_1 \rightarrow R_1$

$$\sim \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Zero Row in Reduced Matrix indicates infinite solutions.

i.e:

$x_1 - \frac{4}{3} x_3 = 0 \Rightarrow x_1 = 0$

$\boxed{\begin{matrix} x_2 = 0 \\ x_3 = 0 \end{matrix}}$ substitute

Also:

As we see, the Rank of Matrix is 2 and the number of unknowns in system are 3.

So: $2 < 3$. According to Condition, the system has non-trivial solutions.

Q#01 Reduce matrix to Normal form and Rank of matrix?

Sol:

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 0 \end{bmatrix}$$

By Row operation:

$R_2 \leftarrow R_2 - 3R_1$; $R_3 - R_1 \rightarrow R_3$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

3)

$3R_3 - R_2 \longrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now: column operation

- $C_2 - 3C_1 \longrightarrow C_2$, $C_4 \leftrightarrow C_2$
- $C_3 - 4C_1 \longrightarrow C_3$, $\frac{C_2}{-6} \rightarrow C_2$
- $C_4 - 3C_1 \rightarrow C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{unit vector}$$

vector

Rank:- Rank $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 0 \end{bmatrix}$

by Reduce matrix to reduced Row Echelon Form.

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- Row Echelon Form,

  - $R_1 \leftrightarrow R_2$
  - $R_2 - \frac{1}{3} \cdot R_1 \to R_2$
  - $R_3 - \frac{1}{3} \cdot R_1 \to R_3$

$$\sim \begin{bmatrix} 3 & 9 & 12 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Now: Reduce Echelon Form,

- $-1 R_3 \to R_3$
- $R_2 - 2R_3 \to R_2$
- $R_1 - 3R_3 \to R_1$
- $R_2 \leftrightarrow R_3$

Reduce matrix to reduced Row Echelon Form.

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- Row echelon form,

- $R_1 \leftrightarrow R_2$
- $R_2 - \frac{1}{3} \cdot R_1 \to R_2$
- $R_3 - 1/3 \cdot R_1 \to R_3$

$$\sim \begin{bmatrix} 3 & 9 & 12 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Now:

Reduce Echelon Form,

- $-1 R_3 \to R_3$
- $R_2 - 2R_3 \to R_2$
- $R_1 - 3R_3 \to R_1$
- $R_2 \leftrightarrow R_3$
- $R_1 - \frac{1}{3} R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Rank} = 2}$

(Rank :> no. of non zero rows)

Result.

Q#04 To show that the given matrix is diagonalizeable?

$$\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

Sol:-

Let;

$$A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

$\therefore$ A matrix A can be diagonalized iff there exists an invertible matrix P and a diagonable matrix D i-e

$$A = PDP^{-1}$$

$\therefore$ Eigenvalues For;

$$A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

• Let A be a (square matrix), η a vector and λ a scalar, that satisfy $A\eta = \lambda\eta$, then λ is called the Eigen value associated with the eigen-vector η of A.

· The eigenvalues of A are the roots of the characteristic equation : $\det(A-\lambda I)=0$

i.e. $$\det\left(\begin{bmatrix}4 & 2 & -2\\ -5 & 3 & 2\\ -2 & 4 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)=0$$

$$= \det\left(\begin{bmatrix}4 & 2 & -2\\ -5 & 3 & 2\\ -2 & 4 & 1\end{bmatrix} - \begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}\right)$$

$$= \det\begin{bmatrix}4-\lambda & 2 & -2\\ -5 & 3-\lambda & 2\\ -2 & 4 & 1-\lambda\end{bmatrix}$$

$$= \begin{vmatrix}4-\lambda & 2 & -2\\ -5 & 3-\lambda & 2\\ -2 & 4 & 1-\lambda\end{vmatrix}$$

$$= (4-\lambda)\left|(3-\lambda)\times(1-\lambda)-8\right|$$
$$-(2)\left|(-5)(1-\lambda)-(2)(-2)\right|$$
$$+(-2)\left|(-5)(4)-(3-\lambda)(-2)\right|$$

P.T.O

$(3-\lambda)(1-\lambda)-8$
$= 3-3\lambda-\lambda+\lambda^2-8$
$= (\lambda^2-4\lambda-5)$

$(-5)(1-\lambda)+4$
$= -5+5\lambda+4$
$= (5\lambda-1)$

$(-20)(-6+2\lambda)$
$= -20+6-2\lambda$
$= (-2\lambda-14)$

$(4-\lambda)(\lambda^2-4\lambda-5) - 2(5\lambda-1) - 2(-2\lambda-14)$

$= 4\lambda^2 - 16\lambda - 20 - \lambda^3 + 4\lambda^2 + 5\lambda - 10\lambda + 2 + 4\lambda + 28$

$= -\lambda^3 + 8\lambda^2 - 17\lambda + 10$

$= -(\lambda-1)(\lambda-2)(\lambda-5)$

Now applying/using zero factor principle i.e;

$ab=0,\ a=0,\ b=0$

$\therefore$

$$\left.\begin{array}{lll} \lambda-1=0 & \Rightarrow & \lambda=1 \\ \lambda-2=0 & \Rightarrow & \lambda=2 \\ \lambda-5=0 & \Rightarrow & \lambda=5 \end{array}\right] \text{solutions}$$

Therefore:

Eigenvalues are;

$\lambda=1,\ \lambda=2,\ \lambda=5$

So, Diagonal Matrix "D" is composed of eigen values i.e;

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Now, the Eigenvectors for

$$\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

∴ Let A be a square matrix, η a vector of λ a scalar that satisfy $A\eta = \lambda\eta$

then, η is an eigenvector of A and λ is the eigen-value associated with it.

1: Eigenvectors for $\lambda = 1$:-

As; $(A - \lambda I) = 0$

$$\therefore A - \lambda I = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{bmatrix}.$$

Now;

$\lambda = 1$

$$\therefore A - \lambda I = \begin{bmatrix} 4-1 & 2 & -2 \\ -5 & 3-1 & 2 \\ -2 & 4 & 1-1 \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}$$

Now; To Solve:

P.T.O

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the matrix;

i.e.

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}$$

Reduce by Echelon Method;

$$R_1 \longleftrightarrow R_2$$

$$\sim \begin{bmatrix} -5 & 2 & -2 \\ 3 & 2 & -2 \\ -2 & 4 & 0 \end{bmatrix}$$

$$R_2 + 3/5 \cdot R_1 \longrightarrow R_2$$

$$\sim \begin{bmatrix} -5 & 2 & -2 \\ 0 & 16/5 & -4/5 \\ -2 & 4 & 0 \end{bmatrix}$$

$$R_3 - 2/5\, R_1 \longrightarrow R_3$$

$$\sim \begin{bmatrix} -5 & 2 & -2 \\ 0 & 16/5 & -4/5 \\ 0 & 16/5 & -4/5 \end{bmatrix}$$

$R_3 - 1R_2 \to R_3$

$$\sim \begin{bmatrix} -5 & 2 & 2 \\ 0 & 16/5 & -4/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, further Reduce by Reduced Row Echelon form;

$\therefore \quad \frac{5}{16} \cdot R_2 \to R_2$

$$\sim \begin{bmatrix} -5 & 2 & 2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 2R_2 \to R_1$

$$\sim \begin{bmatrix} -5 & 0 & 5/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$-\frac{1}{5} R_1 \to R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

Now,

the system associated with

eigenvalue : $\lambda = 1$

$$(A - 1I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduces to

$$\Rightarrow \quad 1x - \tfrac{1}{2} z = 0 \quad , \Rightarrow \quad x = \tfrac{1}{2} z$$

$$1y - \tfrac{1}{4} z = 0 \quad , \Rightarrow \quad y = \tfrac{1}{4} z$$

Plug into $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, i.e

$$\eta = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} z \\ \frac{1}{4} z \\ z \end{bmatrix}, \quad z \neq 0. \quad \text{let } z = 4$$

$$\Rightarrow \quad \eta = \begin{bmatrix} 4/2 \\ 4/4 \\ 4 \end{bmatrix}, \Rightarrow \quad \eta = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

eigenvalue 1, $\lambda = 1$

$$(A - 1I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce:

$\Rightarrow \quad 1x - \frac{1}{2}z = 0 \quad , \Rightarrow \quad x = \frac{1}{2}z$

$\quad\quad 1y - \frac{1}{4}z = 0 \quad , \Rightarrow \quad y = \frac{1}{4}z$

Plug into $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, i.e.

$$\eta = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}z \\ \frac{1}{4}z \\ z \end{bmatrix}, \quad z \neq 0 . \text{ let } z = 4$$

$$\Rightarrow \eta = \begin{bmatrix} 4/2 \\ 4/4 \\ 4 \end{bmatrix}, \Rightarrow \eta = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

2: Eigenvectors for $\lambda = 2$

$$\therefore (A - \lambda I) = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A - 2I = \begin{bmatrix} 4-2 & 2 & -2 \\ -5 & 3-2 & 2 \\ -2 & 4 & 1-2 \end{bmatrix}$$

$\Rightarrow \quad A - 2I = \begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}$

Now,

To solve: $\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Reduce the matrix: $\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}$

by Echelon form:

Swap: $R_1 \longleftrightarrow R_2$

$$\sim \begin{bmatrix} -5 & 1 & 2 \\ 2 & 2 & -2 \\ -2 & 4 & -1 \end{bmatrix}$$

$R_2 + \frac{2}{5} \cdot R_1 \longrightarrow R_2$

$$\sim \begin{bmatrix} -5 & 1 & 2 \\ 0 & \frac{12}{5} & -\frac{6}{5} \\ -2 & 4 & -1 \end{bmatrix}$$

$R_3 - \frac{2}{5} \cdot R_1 \longrightarrow R_3$

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$$\sim \begin{bmatrix} -5 & 1 & 2 \\ 0 & \frac{12}{5} & -\frac{6}{5} \\ 0 & \frac{18}{5} & -\frac{9}{5} \end{bmatrix} \begin{array}{l} R_2 \\ R_3 \end{array}$$

$R_2 \longleftrightarrow R_3$

$$\sim \begin{bmatrix} -5 & 1 & 2 \\ 0 & 18/5 & -9/5 \\ 0 & 12/5 & -6/5 \end{bmatrix}$$

$R_3 - \frac{2}{3} R_2 \longrightarrow R_3$

$$\sim \begin{bmatrix} -5 & 1 & 2 \\ 0 & 18/5 & -9/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, further Reduce the above matrix by Reduced Row Echelon form,

i.e, $\frac{5}{18} \cdot R_2 \longrightarrow R_2$

$$\sim \begin{bmatrix} -5 & 1 & 2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 + 1 \cdot R_2 \longrightarrow R_1$$

$$\sim \begin{bmatrix} -5 & 0 & 5/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-\tfrac{1}{5} \cdot R_1 \longrightarrow R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now,
the system associated with the eigenvalue; $\lambda = 2$

$$(A - 2I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The above matrix reduces to the following system of equations;

$$x - \tfrac{1}{2} z = 0 \quad \Rightarrow \quad x = \tfrac{1}{2} z$$

$$y - \tfrac{1}{2} z = 0, \quad \Rightarrow \quad y = \tfrac{1}{2} z$$

plug into $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

i.e;

$$\eta = \begin{bmatrix} 1/2\, z \\ 1/2\, z \\ z \end{bmatrix}, \quad z \neq 0, \quad \text{let } z = 2$$

$$\Rightarrow \eta = \begin{bmatrix} 2/2 \\ 2/2 \\ 2 \end{bmatrix}$$

$$\Rightarrow \eta = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \rightarrow 2$$

Now;

3) Eigenvectors for $\lambda = 5$,

Solve: $(A - \lambda I)$:

$$\Rightarrow \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A - 5I = \begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix}$$

Now, reduce the matrix by Row Echelon form;

i.e: $\begin{bmatrix} -5 & -2 & 2 \\ -1 & 2 & -2 \\ -2 & 4 & -4 \end{bmatrix}$, $R_1 \leftrightarrow R_2$

$R_2 - \frac{1}{5} \cdot R_1 \rightarrow R_2$

$$\sim \begin{bmatrix} -5 & -2 & 2 \\ 0 & 12/5 & -12/5 \\ -2 & 4 & -4 \end{bmatrix}$$

$R_3 - \frac{2}{5} \cdot R_1 \rightarrow R_3$

$$\sim \begin{bmatrix} -5 & -2 & 2 \\ 0 & 12/5 & -\frac{12}{5} \\ 0 & 24/5 & -24/5 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} -5 & -2 & 2 \\ 0 & 24/5 & -24/5 \\ 0 & 12/5 & -12/5 \end{bmatrix}$$

$R_3 - \frac{1}{2} \cdot R_2 \rightarrow R_3$

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$$\sim \begin{bmatrix} -5 & -2 & 2 \\ 0 & 24/5 & -24/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Now,
further reduce the above matrix by Reduced Row Echelon form;

$$\frac{5}{24} \cdot R_2 \to R_2$$

$$\sim \begin{bmatrix} -5 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 + 2 \cdot R_2 \to R_1$$

$$\sim \begin{bmatrix} -5 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-1/5 \cdot R_1 \to R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The system associated with eigenvalue $\lambda = 5$

$$(A - 5I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This reduces to;

$x = 0$, $\Rightarrow$ $x = 0$

$y - z = 0$, $\Rightarrow$ $y = z$

Plug into, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

i.e, $\eta = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 \\ z \\ z \end{bmatrix}$

$z \neq 0$

Let $z = 1$

$\therefore \eta = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, The eigenvectors corresponding to the eigenvalues in D compose the columns of P;

$$P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Now:

$$P^{-1} = ?$$

$$\therefore \quad P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}^{-1}$$

$\therefore$ Augmented with a 3x3 identity matrix

i.e:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Reduce by Echelon form

$$R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$R_2 - \frac{1}{4} R_1 \longrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1/2 & 3/4 & 0 & 1 & -1/4 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$R_3 - \frac{1}{2} \cdot R_1 \longrightarrow R_3$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{3}{4} & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \end{array}\right]$$

Now, further Reduce by Reduced Row Echelon form;

$-2R_3 \longrightarrow R_3$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{3}{4} & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$R_2 - \frac{3}{4} R_3 \longrightarrow R_2$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$R_1 - 1R_3 \longrightarrow R_1$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 2 & 0 & 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$R_1 - 2R_2 \rightarrow R_1$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 2 & -2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$\frac{1}{4}R_1 \rightarrow R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 3 & 2 & -2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}.$$

Now;

$PDP^{-1} = ?$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

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$$= \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 5 \\ 4 & 4 & 5 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

$$PDP^{-1} = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$, Diagonalized

$\therefore \quad A = PDP^{-1}$ Result