# FINAL TERM PAPER.

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# Differential Equation.

Q1) (i):. $w = \sin(x+ct) + \cos(2x+ct)$

**Given:-**

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \rightarrow (1)$$

Now,

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+ct)\right]$$

$$= \frac{\partial}{\partial t}\left(\sin(x+ct) + \frac{\partial}{\partial t}(\cos(2x+2ct)\right)$$

$$= \frac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct))$$

Now,

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[c\cos(x+ct) - 2c\sin(2x+2ct)\right]$$

$$\boxed{\frac{\partial^2 w}{\partial t^2} = -c^2\sin(x+ct) - 4c^2\cos(2x+ct)}$$

Now,

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$\frac{dw}{dx} = \cos(x+ct) - 2\sin(2x+2ct)$$

$$\frac{d^2w}{dx^2} = \frac{d}{dx}\left[\cos(x+ct) - 2\sin(2x+2ct)\right]$$

$$\frac{d^2w}{dx^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$① \Rightarrow$$

$$-c^2\sin(x+ct) - 4c^2\cos(2x+2ct) = c^2\left[-\sin(x+ct) - 4\cos(2x+2ct)\right]$$

$$-c^2\cancel{\sin}(x+ct) - 4c^2\cancel{\cos}(2x+2ct) = -c^2\cancel{\sin}(x+ct) - 4c^2\cancel{\cos}(2x+2ct)$$

$0 = 0$ (Satisfied).

ii):. $w = \tan(2x + ct)$.

## Solution:.

Now, $$\frac{\partial w}{\partial t} = c\sec^2(2x + ct)$$

& $$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(c\sec^2(2x+ct)\right)$$

$$= c^2 \cdot 2\sec^2(2x + ct)\tan(2x+ct)$$

Now,

$$\frac{\partial w}{\partial x} = 2\sec^2(2x+ct)$$

$$\frac{\partial^2 w}{\partial x^2} = 4\sec^2(2x+ct)\tan(2x+ct)$$

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$$4c^2\cancel{\sec^2}(2x+ct)\tan(2x+ct) = 4c^2\cancel{\sec^2}(2x+ct)\tan(2x+ct)$$

$$0 = 0 \quad (\text{Satisfied})$$

QNo: 2 . Given Function.

$$f(x) = \begin{cases} x; & -\pi < x \leq 0 \\ 2x & 0 \leq x < \pi \end{cases}$$

## Solution:.

We have to find the fourier coefficients, $a_0, a_n$ & $b_n$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx - \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$a_0 = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{\pi}\right]_{0}^{\pi}$$

$$a_0 = \frac{1}{\cancel{\pi}}\left\{0 - \frac{\pi^{\cancel{x}}}{2}\right\} + \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^{\cancel{x}}}{\cancel{2}} - 0\right]$$

$$\boxed{a_0 = -\frac{\pi}{2} + \pi = \pi/2} \longrightarrow (i).$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos x\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(x\cos x\,dx) + \frac{1}{\pi}\int_{0}^{\pi}(2x\cos nx)\,dx$$

$$a_n = \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$a_n = \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos nx}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos nx}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{1 - (-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

So,

$$\boxed{a_n = \begin{cases} \frac{-2}{\pi n^2} & ; \text{ if } n \text{ is odd} \\ 0 & ; \text{ if } n \text{ is even} \end{cases}} \longrightarrow (2).$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx.$$

$$b_n = \frac{1}{\pi}\left[x\left(\frac{-\cos x}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$b_n = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] = -\frac{3\cos n\pi}{n}$$

$$= \frac{3(-1)^{n+1}}{n} \longrightarrow (3).$$

So, the required fourier series is.

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$$

QNo: 3/:- $y'' - 4y' + 13y = 8\sin 3x$

$y(0) = 1$ & $y'(0) = 2$

Solution:-

We have to find $y = y_c + y_p$

For $y_c$ the characteristic (auxilary Eqn) Eqn is:-

use Quadratic formula,

$m^2 - 4m + 13 = 0$ $a = 1, b = -4, c = 13$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$m = \frac{4 \pm \sqrt{16-52}}{2}$ $m = \frac{4 + 6i}{2}$

$\rightarrow m = 2 \pm 3i$ ; $\alpha = 2$ & $\beta = 3$.

So, $y_c = e^{2x}\{C_1 \cos 3x + C_2 \sin 3x\}$.

For $y_p$,

let

$y_p = \text{Imag}\left(\frac{1}{m^2 - 4m + 13} 8e^{3ix}\right)$

$y_p = 8\,\text{Imag}\, \frac{e^{3ix}}{(3i)^2 - 4(3i) + 13}$

$y_p = 8\,\text{Imag}\, \frac{e^{3ix}}{-9 - 12i + 13}$

$$y_p = 8\, \text{Imag}\ \frac{e^{3ix}}{4-12i}$$

$$y_p = 2\, \text{Imag}\ \frac{e^{3ix}}{(1-3i)} \times \frac{(1+3i)}{(1+3i)}$$

$$y_p = 2\, \text{Imag} \cdot \frac{(1+3i)(e^{3ix})}{(1)^2-(3i)^2}$$

$$y_p = 2\, \text{Imag} \cdot \frac{(1+3i)(e^{3ix})}{10}$$

$$y_p = \frac{2}{10}\left(\text{Imag} \cdot (1+3i)(\cos 3x + i \sin 3x)\right)$$

$$y_p = \frac{2}{10}(\sin 3x + 3\cos 3x)$$

So, the general solution is,

$$y = y_c + y_p.$$

$$y = c_1 e^{2x}\cos 3x + c_2 e^{2x}\sin 3x + \frac{2}{10}(\sin 3x + 3\cos 3x)$$

Now use the initial condition $y(0) = 1$

$$y(0) = c_1 e^{(0)}\cos(0) + c_2 e^{(0)}\sin(0) + \frac{2}{10}(\sin(0) + 3\cos(0))$$

$$1 = c_1(1) + 0 + 0 + \frac{2}{10}(3(1))$$

$$1 = c_1 + \frac{6}{10}$$

$$c_1 = 1 - \frac{6}{10}, \quad c_1 = \frac{4}{10}, \quad \boxed{c_1 = 2/5}$$

Again use the another Initial Condition

$$y'(0) = 2$$

So,

$$y' = c_1 2e^{2x} \cos 3x + c_1 e^{2x}(-3\sin 3x) + c_2 2e^{2x} \sin 3x +$$

$$c_2 e^{2x}(3\cos 3x) + \frac{2}{10}(\cos 3x - 3\sin 3x)$$

$$y'(0) = c_1 2e^{(0)} \cos(0) + c_1 e^{(0)}(-3\sin(0)) + c_2 2e^{(0)} \sin(0)$$

$$+ c_2 e^{(0)}(3\cos(0)) + \frac{2}{10}(\cos(0) - 3(\sin(0))$$

$$2 = 2c_1 + 0 + 0 + c_2 3(1) + \frac{2}{10}(1 - 3(0))$$

$$2 = 2c_1 + 3c_2 + \frac{2}{10}$$

$$2 = 2\left(\frac{2}{5}\right) + 3c_2 + \frac{2}{10} \qquad \boxed{\text{use } c_1 = 2/5}$$

$$c_2 = \frac{1}{3}\left(2 - \frac{4}{5} - \frac{2}{10}\right)$$

$$c_2 = \frac{1}{3}\left(\frac{2-8-2}{10}\right)$$

$$\boxed{c_2 = \frac{1}{3}}$$

So the General Solution is,

$$y = \frac{2}{5} e^{2x} \cos 3x + \frac{1}{13} e^{2x} \sin 3x + \frac{2}{10} [\sin 3x + 3\cos 3x]$$

be the Required Solution.

**Q4):- Solve.**

$$(D^2 - DD')z = \cos x \cos 2y.$$

**Solution:-**

The Auxiliary equation is,

$m^2 - m = 0 \Rightarrow m = 0, m = 1$

Hence the complementary function is given by

$z_c = f_1(y) + f_2(y+x)$.

For the particular integral we have $z_p = \frac{1}{D^2 - DD'} \cdot \cos x \cos 2y$

$$z_p = \frac{1}{2} \cdot \frac{1}{D^2 - DD'} [\cos(x-2y) + \cos(x-2y)]$$

$$z_p = \frac{1}{2} \left[ \frac{1}{D^2 - DD'} \cos(x+2y) + \frac{1}{D^2 - DD'} \cos(x-2y) \right]$$

$$= \frac{1}{2} \left[ \frac{1}{-1+2} \cos(x+2y) + \frac{1}{-1-2} \cos(x-2y) \right]$$

$$= \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Hence, the complete equation is given by:

$$z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$ Answer.