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Assignment # 02

Q#01:-

## The Cauchy Euler Equation

$$\textcircled{1} \quad x^3 y''' + 2x^2 y' + 2y = 10x + \frac{10}{x}$$

Solution:-

$$x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{dy}{dx} + 2y = 10x + 10x^{-1}$$

$$x^3 D^3 y + 2x^2 D^2 y + 2y = 10x + 10x^{-1}$$

$$(x^3 D^3 + 2x^2 D^2 + 2) y = 10x + 10x^{-1} \rightarrow \textcircled{1}$$

$$\text{let } x = e^t \Rightarrow t = \ln x$$

$$x D = \Delta$$

$$x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$$

$$x^3 D^3 = \Delta(\Delta - 1)(\Delta - 2)$$

substituting into eq(1)

$$(\Delta - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta) + 2)y = 10x + 10x^{-1}$$

$$(\Delta^3 - \Delta^2 + 2)y = 10x + 10x^{-1}$$

$$(m^3 - m^2 + 2)y = 10e^t + \frac{10}{e^t}$$

using synthetic division

$$\begin{array}{c|cccc} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & 2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

$$\Delta^2 - 2\Delta + 2 = 0$$

Now using Quadratic Formula

$$a = 1, b = -2, c = 2$$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta = \frac{-(-2) \pm \sqrt{-2^2 - 4(1)(-2)}}{2(1)}$$

$$\Delta = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\Delta = \frac{2 \pm \sqrt{-4}}{2}$$

$$\Delta = \frac{2 \pm \sqrt{-1} \sqrt{4}}{2}$$

$$\Delta = \frac{2 \pm 2i}{2}$$

$$\Delta = \frac{2(1 \pm i)}{2}$$

$$\Delta = 1 \pm i$$

Since roots are complex

$$y_c = e^{-x} (c_1 \cos t + c_2 \sin t)$$

Now Particular Integration

$$y_p = \frac{1}{D^3 - D^2 + 2} \cdot 10e^t + \frac{1}{D^3 - D^2 + 2} \cdot 10e^{-t}$$

$$= \frac{10e^t}{(1)^3 - (1)^2 + 2} + \frac{10e^{-t}}{(1)^3 - (1)^2 + 2}$$

$$= \frac{5}{2} 10e^t + \frac{5}{2} 10e^{-t}$$

$$= 5e^t + 5e^{-t}$$

$$y_p = 5e^t + 5e^{-t}$$

General Solution:

$$y = y_c + y_p$$

$$y = e^{-x} (c_1 \cos t + c_2 \sin t) + 5e^t + 5e^{-t}$$

Put  $e^t = x$  and  $t = \ln x$

$$y = e^{-x} (c_1 \ln x + c_2 \sin \ln x) + 5e^x + 5e^{-x}$$

Q# 23

$$\frac{x^3 d^3 y}{dx} + 4x^2 \frac{d^2 y}{dx} - 5x \frac{dy}{dx} - 15y = x^4$$

Sol:-

$$\text{Let } \frac{d}{dx} = D$$

$$x^3 D^3 y + 4x^2 D^2 y - 5x D y - 15y = x^4$$

$$(x^3 D^3 + 4x^2 D^2 - 5x D - 15)y = x^4$$

$$\text{let } x = e^t \Rightarrow t = \ln x$$

$$xD = D$$

$$x^2 D^2 = \Delta(\Delta-1) = D^2 - D$$

$$x^3 D^3 = \Delta(\Delta-1)(\Delta-2) = D^3 - 3D^2 + 2D$$

Now substituting:-

$$(x^3 D^3 + 4x^2 D^2 - 5x D - 15)y = x^4$$

$$(\Delta^3 - 3\Delta^2 + 2\Delta + 4(\Delta^2 - \Delta) - 5(\Delta) - 15)y = e^{4t}$$

$$\cancel{(\Delta^3 + \Delta^2 + \Delta)}$$

$$(\Delta^3 + \Delta^2 - 7\Delta - 15)y = e^{4t}$$

synthetic division

5	1	+1	-7	-15	
		3	12	15	
	1	4	5	0	

$$\Delta^2 + 4\Delta + 5 = 0$$

Quadratic formula

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{(-4) \pm \sqrt{4^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{(-4) \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= \frac{\cancel{(-2)} \pm (-2 \pm i)}{\cancel{2}}$$

$$y_c = e^{3x} (C_1 \cos t + C_2 \sin t)$$

For  $y_p = ?$

$$y_p = \frac{1}{\Delta^3 + \Delta^2 - 7\Delta - 15} \cdot e^{4t}$$

$$= \frac{1}{(4)^3 + (4)^2 - 7(4) - 15} \cdot e^{4t}$$

$$= \frac{1}{80 - 43} \cdot e^{4t}$$

E...

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$$y_p = \frac{1}{37} e^{4t}$$

Hence

$$y = y_c + y_p$$

$$y = (C_1 \cos t + C_2 \sin t) + \frac{1}{37} e^{4t}$$

Again put  $t = \ln x$  and  $x = \ln x$

$$y = e^{3x} (C_1 \cos \ln x + C_2 \sin \ln x) + \frac{1}{37} e^{4t}$$

P.S.  $x^2 y'' + 2xy' - 6y = 10x^2 \rightarrow (A) \quad y(1) = 1, \quad y'(1) = -6.$

S.P. A.H. Equation:

Let  $x^2 y'' + 2xy' - 6y = 0 \rightarrow (B)$

Let  $y = x^m$

Diff. w.r.t.  $x \Rightarrow y' = m x^{m-1}$

Again Diff. w.r.t.  $x \Rightarrow y'' = (m-1)(m) x^{m-2}$

putting values in (B).

$x^2 ((m^2 - m) x^{m-2}) + 2x (m x^{m-1}) - 6x^m = 0$

$(m^2 - m) x^m + 2m x^m - 6x^m = 0$

$(m^2 - m + 2m - 6) x^m = 0$

$x^m \neq 0$

Then  $m^2 + m - 6 = 0.$

$m^2 + 3m - 2m - 6 = 0$

$m(m+3) - 2(m+3) = 0$

$(m-2)(m+3) = 0$

$m-2 = 0$

$m+3 = 0$

$m = 2, \quad m = -3.$

$y_c = C_1 x^2 + C_2 x^{-3} \rightarrow (A)$

For Non-homogenous Eq. We use variation of parameter method:

$x^2 y'' + 2xy' - 6y = 10x^2$

Dividing by  $x^2$

$y'' + \frac{2}{x} y' - \frac{6}{x^2} = 10$

$f(x) = 10$

$y_1 = x^2$



Now

Let

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$$y_p = u_1 x^2 + u_2 x^{-3} \rightarrow (**)$$

By using variation of p

$$W = \begin{vmatrix} x^2 & x^{-3} \\ 2x & -3x^{-4} \end{vmatrix} = \frac{-3x^{-2} - 2x^{-2}}{-5x^{-2}}$$

$$W_1 = \begin{vmatrix} 10 & x^{-3} \\ 0 & -3x^{-4} \end{vmatrix} = \frac{-30x^{-4}}{-5x^{-2}}$$

$$W_2 = \begin{vmatrix} x^2 & 10 \\ 2x & 0 \end{vmatrix} = \frac{-20x}{-5x^{-2}}$$

$$u_1' = \frac{W_1}{W} = \frac{-30x^{-4}}{-5x^{-2}} = 6x^{-2}$$

$$u_2' = \frac{W_2}{W} = \frac{-20x}{-5x^{-2}} = 4x^3$$

$$u_1' = 6x^{-2}$$

Taking  $\int$  on b/sides

$$\int du_1 = \int 6x^{-2} dx$$

$$u_1 = \frac{6x^{-1}}{-1}$$

$$u_1 = -6x^{-1}$$

$$u_2' = 4x^3$$

Taking  $\int$  on b/sides

$$\int du_2 = \int 4x^3 dx$$

$$u_2 = \frac{4x^4}{4}$$

$$u_2 = x^4$$

putting the values in (\*\*)

$$y_p = (-6x^{-1})(x^2) + (x^4)(x^{-3})$$

$$= -6x + x$$

$$y_p = -5x \rightarrow (***)$$

For

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$$y = y_c + y_p \rightarrow \text{xxxx}$$

put (x) & (xxx) in (xxxx)

We get:

$$y = c_1 x^2 + c_2 x^{-3} - 5x \quad \rightarrow \textcircled{C}$$

Now we need to find the values of  $c_1$  &  $c_2$ .

For this:

put  $y(1) = 1$  in Eq (C).

$$1 = c_1 (1)^2 + c_2 (1)^{-3} - 5(1)$$

$$1 = c_1 + c_2 - 5$$

$$c_1 + c_2 = 6 \rightarrow \textcircled{a}$$

Diff: Eq (C) w.r.t to 'x'.

$$y' = 2c_1 x - 3c_2 x^{-2} - 5$$

put  $y'(1) = -6$ .

$$-6 = 2c_1 (1) - 3c_2 (1) - 5$$

$$-6 = 2c_1 - 3c_2 - 5$$

$$2c_1 - 3c_2 = -1$$

$$2c_1 = +3c_2 - 1$$

$$c_1 = \frac{3}{2} c_2 - \frac{1}{2} \rightarrow \textcircled{b}$$

put Eq (b) in Eq (a)

Q4.

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$$\left(\frac{3}{2}c_2 - \frac{1}{2}\right) + c_2 = 6$$

xing by 2

$$3c_2 - 1 + 2c_2 = 12$$

$$5c_2 = 12 + 1$$

$$c_2 = \frac{13}{5} \rightarrow \text{(d) put in (b)}$$

$$c_1 = \frac{3}{2} \left(\frac{13}{5}\right) - \frac{1}{2}$$

$$= \frac{39}{10} - \frac{1}{2}$$

$$c_1 = \frac{17}{5} \rightarrow \text{(e)}$$

put Eq (d) & (e) in Eq (c)

$$y = \frac{17}{5}x^2 + \frac{13}{5}x^3 - 5x$$

→ The Required General Solution.

Q4:

$$x^2 y'' + 7xy' + 5y = x^5 \rightarrow \textcircled{A}$$

$y(0) = 0, \quad y(1) = 0$

Sol:

A.H - Eq:

$$x^2 y'' + 7xy' + 5y = 0 \rightarrow \textcircled{1}$$

let

$$y = x^m$$

$$y' = m x^{m-1}$$

$$y'' = m(m-1) x^{m-2}$$

put in  $\textcircled{1}$

$$x^2 (m(m-1) x^{m-2}) + 7x (m x^{m-1}) + 5x^m = 0$$

$$(m^2 - m) x^m + 7m x^m + 5x^m = 0$$

$$(m^2 - m + 7m + 5) x^m = 0$$

let  $x^m \neq 0$

Then

$$m^2 + 6m + 5 = 0$$

$$m^2 + 5m + m + 5 = 0$$

$$m(m+5) + 1(m+5) = 0$$

$$(m+1)(m+5) = 0$$

$$m = -1, \quad m = -5$$

$$y_c = c_1 x^{-1} + c_2 x^{-5} \rightarrow \textcircled{a}$$

$\textcircled{A} \Rightarrow$

$$x^2 y'' + 7xy' + 5y = x^5$$

$$y'' + \frac{7}{x} y' + \frac{5}{x^2} y = x^3 \rightarrow \textcircled{2}$$

$$y_p = u_1 x^{-1} + u_2 x^{-5} \rightarrow \textcircled{x}, \quad f(x) = x^3$$

$$W_1 = \begin{vmatrix} x^{-1} & x^{-5} \\ -x^{-2} & -5x^{-6} \end{vmatrix} = -5x^{-7} + x^{-7} = -4x^{-7} \rightarrow \textcircled{i}$$

$$W_{11} = \begin{vmatrix} x^3 & x^{-5} \\ 3x^2 & -5x^{-6} \end{vmatrix} = -5x^{-3} - 3x^{-3} \Rightarrow -8x^{-3} \rightarrow \textcircled{ii}$$

$$W_2 = \begin{vmatrix} x^{-1} & x^2 \\ -x^{-2} & 3x^2 \end{vmatrix} = 3x + x = 4x \rightarrow \textcircled{iii}$$

$$u_1' = \frac{w_1}{w}$$

putting values

$$u_1' = \frac{7 \cdot 2x^{-3}}{x \cdot x^{-7}}$$

$$= 2x^{-3} \cdot x^7$$

$$u_1' = 2x^4$$

Taking  $\int$  on b/s

$$\int u_1' = \int 2x^4 dx$$

$$u_1 = 2 \frac{x^5}{5}$$

$$\boxed{u_1 = \frac{2}{5} x^5}$$

$$u_2' = \frac{w_2}{w}$$

put

$$u_2' = \frac{1 \cdot x}{-1 \cdot x^{-7}}$$

$$u_2' = -x^8 \cdot x^7$$

$$u_2' = -x^8$$

Tak  $\int$  on b/c.

$$\int u_2' = - \int x^8 dx$$

$$u_2 = - \frac{x^9}{9}$$

$$\boxed{u_2 = -\frac{x^9}{9}}$$

put in (\*)

$$(*) \Rightarrow y_p = \left( \frac{2}{5} x^5 \right) (x) + \left( -\frac{x^9}{9} \right) (x^{-5})$$

$$y_p = \frac{2}{5} x^4 - \frac{x^9}{9} x^{-5}$$

$$y_p = \frac{2}{5} x^4 - \frac{x^4}{9}$$

$$= x^4 \left( \frac{2}{5} - \frac{1}{9} \right)$$

$$y_p = \left( \frac{18 - 5}{45} \right) x^4$$

$$y_p = \frac{13}{45} x^4 \rightarrow (**)$$

The General Solution is

$$y = y_c + y_p \rightarrow (***)$$

putting  $(**)$  &  $(a)$  in  $(***)$

$$y = C_1 x^{-1} + C_2 x^{-5} + \frac{13}{45} x^4 \rightarrow (B)$$

~~put  $y(0) = 2$  in  $(B)$~~

$$\boxed{y_1 = C_1 x^{-1} + C_2 x^{-5} + \frac{13}{45} x^4}$$

Q# 5:-

$$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2$$

Sol:

$$(x+1)^2 \frac{d^2 y}{dx^2} - 3(x+1) \frac{dy}{dx} + 4y = x^2$$

$$\Rightarrow (x+1)^2 \frac{d^2}{dx^2} - 3(x+1) \frac{d}{dx} + 4) y = x^2$$

$$\Rightarrow [(x+1)^2 \Delta^2 - 3(x+1) \Delta + 4] y = x^2 \rightarrow \textcircled{1}$$

Put  $(x+1)\Delta = \Delta \Rightarrow (x+1)^2 \Delta^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

$x = e^t$  in equ  $\textcircled{1}$

$$\Rightarrow [\Delta^2 - \Delta - 3\Delta + 4] y = e^{2t}$$

$$\Rightarrow [\Delta^2 - 4\Delta + 4] y = e^{2t}$$

$$\Rightarrow [\Delta^2 - 4\Delta + 4] y = e^{2t}$$

For  $y_c$  we find the roots

$$\Delta^2 - 4\Delta + 4 = 0$$

$$\Delta^2 - 2\Delta - 2\Delta + 4 = 0$$

$$\Delta(\Delta-2) - 2(\Delta-2) = 0$$

$$(\Delta-2)(\Delta-2) = 0$$

$$(\Delta-2) = 0, (\Delta-2) = 0$$

$$\Delta = 2, \Delta = 2$$

So roots are equal and ~~repeated~~ repeated

The General Solution are

$$y = (c_1 + c_2 x)^{1x}$$

$$y = (c_1 + c_3 x)^{2x}$$

For  $y_p = ?$

$$y_p = \frac{1}{D^2 - 4D + 4} \quad \left| \quad (2)^2 - 4(2) + 4 \right.$$

$$y_p = \frac{2}{2D - 4} e^{2t} \quad \left| \quad \Rightarrow 0 \right.$$

If we put 2

$$2D - 4 \Rightarrow 2(2) - 4 = 0$$

We take again derivative

$$y_p = \frac{2}{2} e^{2t}$$

$$y = (c_1 + c_2 x)^{2t} + e^{2t}$$