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Paper:→ Differential Equation
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Exam:→ Final Term (Spring-2020)

Q NO:→ (01)

Solution

(1) ⇒ $w = \sin(x+ct) + \cos(2x+2ct)$

Given $\dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2}$

Now $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right]$

$= \dfrac{\partial}{\partial t}\left(\sin(x+ct)\right) + \dfrac{\partial}{\partial t}\left(\cos(2x+2ct)\right)$

$\dfrac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct)$

Now $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial t}\left[c\cos(x+ct) - 2c\sin(2x+2ct)\right]$

P.T.O

$$\boxed{\frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x+ct) - 4c^2 \cos(2x+2ct)}$$

Now $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct) + \cos(2x+2ct)\right]$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - 2\sin(2x+2ct)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct) - 2\sin(2x+2ct)\right]$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

(1) ⇒

$$-c^2\sin(x+ct) - 4c^2\cos(2x+2ct) = c^2\left[-\sin(x+2ct) - 4\cos(2x+2ct)\right].$$

$$\cancel{-c^2\sin(x+ct)} \cancel{-4c^2\cos(2x+2ct)} = \cancel{-c^2\sin(x+ct)} \cancel{-4c^2\cos(2x+2ct)}$$

$$0 = 0 \text{ (Satisfied)}$$

P.T.O

(ii)⇒ $w = \tan(2x+ct)$

Now $\dfrac{\partial w}{\partial t} = c\,Sec^2(2x+ct)$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(c\,Sec^2(2x+ct)\right)$$

$$= C^2 \cdot 2Sec^2(2x+ct)\tan(2x+ct)$$

Now $\dfrac{\partial w}{\partial x} = 2Sec^2(2x+ct)$

$$\frac{\partial^2 w}{\partial x^2} = 4Sec^2(2x+ct)\tan(2x+ct)$$

(1)⇒

$$4c^2 \cancel{Sec^2(2x+ct)\tan(2x+ct)} = 4c^2 \cancel{Sec^2(2x+ct)\tan(2x+ct)}$$

$0 = 0$ Satisfied

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P.T.O

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Q NO:- (02)

Solution

Given function is

$$F(x) = \begin{cases} x; & -\pi < x \le 0 \\ 2x; & 0 \le x \le \pi \end{cases}$$

We have to find the Fourier Co-efficients, a_0, a_n & b_n.

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left[0 - \frac{\pi^{\cancel{2}}}{2}\right] + \frac{2}{\pi}\left[\frac{\pi^2}{\cancel{2}} - 0\right]$$

$$\boxed{a_0 = -\frac{\pi}{2} + \pi = \frac{\pi}{2}} \rightarrow (1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx$$

P.T.O

$$\Rightarrow = \frac{1}{\pi}\int_{-\pi}^{0}(x\cos nx\,dx) + \frac{1}{\pi}\int_{0}^{\pi}(2x\cos nx)\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

So,

$$\boxed{a_n \begin{cases} \dfrac{-2}{\pi n^2} ; & \text{if } n \text{ is add} \\ 0 ; & \text{if } n \text{ is even} \end{cases}} \rightarrow (2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

P.T.O

$$\Rightarrow \boxed{b_n = \frac{1}{\pi}\left[-\frac{\pi \cos n\pi}{n}\right] + \frac{2}{\pi}\left[-\frac{\pi \cos n\pi}{n}\right] = -\frac{3\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}} \rightarrow (3)$$

So the Required fourier Series is: $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

Again use the another Initial Condition

$$y'(0) = 2$$

So,

$$y' = C_1 2e^{2x}\cos 3x + C_1 e^{2x}(-3\sin 3x) + C_2 2e^{2x}\sin 3x + C_2 e^{2x}(3\cos 3x) + \frac{2}{10}(\cos 3x - 3\sin 3x)$$

$$y'(0) = C_1 2e^{(0)}\cos(0) + C_1 e^{(0)}(-3\sin(0)) + C_2 2e^{(0)}\sin(0) + C_2 e^{(0)}(3\cos(0)) + \frac{2}{10}(\cos(0) - 3(\sin(0))$$

P.T.O

$$\Rightarrow \quad 2 = 2c_1 + 0 + 0 + C_2\,3(1) + \frac{2}{10}\left(1 - 3(0)\right)$$

$$2 = 2c_1 + 3C_2 + \frac{2}{10}$$

$$2 = 2(2/5) + 3C_2 + \frac{2}{10}$$

use $C_1 = 2/5$

$$\frac{1}{3}\left(2 - \frac{4}{5} - \frac{2}{10}\right) = C_2 \Rightarrow \boxed{C_2 = \frac{1}{3}\left(\frac{20 - 8 - 2}{10}\right) = \frac{1}{3}}$$

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Q NO:→ (03)

Solution

Given $\quad y'' - 4y' + 13y = 8\sin 3x$

We have to find $y = y_c + y_p$

For y_c the Characteristic (auxilary Eqn) Eqn is:

$$m^2 - 4m + 13 = 0$$

$$\Rightarrow m = \frac{4 \pm \sqrt{16 - 52}}{2} \Rightarrow m = \frac{4 \pm 6i}{2}$$

$$\Rightarrow m = 2 \pm 3i \; ; \; \alpha = 2 \;\&\; \beta = 3$$

So, $\quad y_c = e^{2x}\{c_1 \cos 3x + c_2 \sin 3x\}$

P.T.O

$\Rightarrow$ y_p let

$$y_p = \text{Imag.}\left(\frac{1}{m^2-4m+13}\, 8e^{3ix}\right)$$

$$= 8\,\text{Imag}\ \frac{e^{3ix}}{(3i)^2-4(3i)+13}$$

$$= 8\,\text{Imag}\ \frac{e^{3ix}}{-9-12i+13}$$

$$= 8\,\text{Imag}\ \frac{e^{3ix}}{4-12i}$$

$$y_p = 2\,\text{Imag.}\ \frac{e^{3ix}}{(1-3i)} \times \frac{(1+3i)}{(1+3i)}$$

$$y_p = 2\,\text{Imag}\ \frac{(1+3i)(e^{3ix})}{(1)^2-(3i)^2}$$

$$y_p = 2\,\text{Imag.}\ \frac{(1+3i)(e^{3ix})}{10}$$

$$y_p = \frac{2}{10}\left(\text{Imag}(1+3i)(\cos 3x + i\sin 3x)\right)$$

$$y_p = \frac{2}{10}\left(\sin 3x + 3\cos 3x\right)$$

So the general Solution is

$$y = y_c + y_p$$

$$y = C_1 e^{2x}\cos 3x + C_2 e^{2x}\sin 3x + \frac{2}{10}\left(\sin 3x + 3\cos 3x\right)$$

P.T.O

→ Now use the Initial Condition $y(0) = 1$

$$y(0) = C_1 e^{(0)} \cos(0) + C_2 e^{(0)} \sin(0) + \frac{2}{10}\left(\sin(0) + 3\cos(0)\right)$$

$$1 = C_1 (1) + 0 + 0 + \frac{2}{10}\left(3(1)\right)$$

$$1 = C_1 + \frac{6}{10} \Rightarrow \boxed{C_1 = 1 - \frac{6}{10} = \frac{4}{10} = \frac{2}{5}}$$

Again use the another Initial Condition

So $y'(0) = 2$

$$y' = C_1 2 e^{2x} \cos 3x + C_1 e^{2x} (-3\sin 3x)$$
$$+ C_2 2 e^{2x} \sin 3x + C_2 e^{2x} (3\cos 3x)$$
$$+ \frac{2}{10}\left(\cos 3x - 3\sin 3x\right)$$

$$y'(0) = C_1 2 e^{(0)} \cos(0) + C_1 e^{(0)}(-3\sin(0))$$
$$+ C_2 2 e^{(0)} \sin(0) + C_2 e^{(0)} (3\cos(0))$$
$$+ \frac{2}{10}\left(\cos(0) - 3(\sin(0)\right).$$

$$2 = 2C_1 + 0 + 0 + C_2 3(1) + \frac{2}{10}\left(1 - 3(0)\right)$$

$$2 = 2C_1 + 3C_2 + \frac{2}{10}$$

$$2 = 2\left(\frac{2}{5}\right) + 3C_2 + \frac{2}{10}$$

$$\frac{1}{3}\left(2 - \frac{4}{5} - \frac{2}{10}\right) = C_2 \Rightarrow \boxed{C_2 = \frac{1}{3}\left(\frac{20 - 8 - 2}{10}\right) = \frac{1}{3}}$$

P.T.O

⇒ So the general Solution is

$$y = \frac{2}{5} e^{2x} \cos 3x + \frac{1}{3} e^{2x} \sin 3x + \frac{2}{10}\left[\sin 3x + 3\cos 3x\right]$$

is the Required Solution.

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Q NO:→ (04)

Solution

$$(D^2 - DD')z = \cos x \cos 2y$$

The auxiliary equation is

$$m^2 - m = 0 \Rightarrow m = 0, \; m = 1$$

Hence the Complementry function is given by $z_c = f_1(y) + f_2(y + x)$

For the particular Integral, we have

$$z_p = \frac{1}{D^2 - DD'} \cdot \cos x \cos 2y$$

P.T.O

$$\Rightarrow \frac{1}{2} \cdot \frac{1}{D^2 - DD'} \left[\cos(x - 2y) + \cos(x - 2y) \right]$$

$$= \frac{1}{2} \left[\frac{1}{D^2 - DD'} \cos(x + 2y) + \frac{1}{D^2 - DD'} \cos(x - 2y) \right]$$

$$= \frac{1}{2} \left[\frac{1}{-1+2} \cos(x + 2y) + \frac{1}{-1-2} \cos(x - 2y) \right]$$

$$= \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y)$$

Hence the Complete Solution is given by

$$Z = f_1(y) + f_2(y + x) + \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y)$$