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SUBJECT # Differential Equation

ASSIGNMENT # 1

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Q1) $x^3y''' + 2x^2y' + 2y - 10x + 10/x.$

Solution

$$x^3 \frac{d^3y}{d^3x} + 2x^2\frac{dy}{dx} + 2y = 10x - 10x^{-1}$$

$$x^3D^3y + 2x^2D^3 + 2y = 10x + 10x^{-1}$$

$$y(x^3D^3 + 2x^2D + 8) = 10x + 10x^{-1} \quad — \text{(1)}$$

let $x = e^t \Rightarrow t = \ln x$

$$xD = D$$

$$x^2D^2 = D(\Delta - 1) = \Delta^2 - \Delta$$

$$x^3D^2 = D(\Delta - 1) = (\Delta - 2)$$

Subtituting into equ (i)

$$(D - 3\Delta^2 + 2\Delta + 2\,(\Delta^2 - \Delta + 2)y = 10x + 10x^{-1}$$

$$(\Delta^3 - \Delta^2 + 2)y = 10x + 10x^{-1}$$

$$(m^3 - m^2 + 2)y = 10e^t + 10/e^t$$

Using synthetic division.

~~-1~~	~~1~~	~~-1~~	~~0~~	~~2~~
		~~-1~~	~~2~~	~~-2~~
	~~1~~			~~$\boxed{0}$~~

-1	1	-1	0	2
		-1	2	-
	1	-2	2	

$\Delta^3 - 2\Delta + 2 = 0$

Now using quadratic formula.

$a = 1, \quad b = -2, \quad c = 2$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta = \frac{-(-2) \pm \sqrt{-2^2 - 4(1)(2)}}{2(1)}$$

$$\Delta = \frac{2 \pm \sqrt{-4}}{2}$$

$$\Delta = \frac{2 \pm \sqrt{-1} \times \sqrt{4}}{2}$$

$$\Delta = \frac{2 + 2i}{2}$$

$$\Delta = \frac{2(1 \pm i)}{2}$$

$$\Delta = 1 \pm i$$

Since root are Complex.

$y = e^{-x} (C_1 \operatorname{Cos} t + C_2 \operatorname{Sin} t).$

Now particular integration

$$yp = \frac{1}{D^3+D^2+2} - 10e^t - \frac{1}{D^3+D^2+2} - 10/e^t$$

$$= \frac{10e^t}{(1)^3-(1)^2+2} + \frac{10e^{-t}}{(1)^3-(1)^2+2}$$

$$= \frac{\cancel{10}^{5}e^t}{\cancel{2}} + \frac{\cancel{10}^{5}e^t}{\cancel{2}}$$

$$= 5e^t + 5e^t$$

$$yp = 5e^t + 5e^t$$

General solution

$$y = yc - yp$$

$$y = e^{-x}(c_1 \cos t + c_2 \sin t) + 5e^t + 5e^t$$

put $e^t = x$ & $t = \ln x$

$$y = \cancel{e^{x}(c_1 \cos t \ln t}$$

$$y = e^{-x}(c_1 \ln x + c_2 \sin \ln x) + 5e^{x} + 5e^{-}$$

Q_2 :-

$$\frac{x^3 d^3y}{dx} + \frac{4x^2 d^2y}{dx} - \frac{5x\,dy}{dx} - 15 = x^4.$$

Solution :-

Let $\frac{d}{dx} = D$

$$x^3 D^3 y + 4x^2 D^2 y - 5x\,Dy - 15y = x^4$$

$$(x^3 D^3 + 4x^3 D^2 - 5xD - 15)y = x^4$$

let $x = e^t \Rightarrow t = \ln x$

$$xD = D$$

$$x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$$

$$x^3 D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$$

Now subtituting

$$(x^3 D^3 + 4x^2 D^2 - 5xD - 15)y = x^4$$

$$(\Delta^3 - 3\Delta^2 + 2\Delta + 4(\Delta^2 - D) - 5(\Delta) - 15)y = x^4.$$

(5)

$$(\Delta^3 + \Delta^2 + 7\Delta - d)\,y = e^{xt}$$

synthetic division.

$$\begin{array}{c|cccc}
5 & 1 & +1 & -7 & -15 \\
 & & 3 & 12 & 15 \\
\hline
 & 1 & 4 & 5 & \boxed{0}
\end{array}$$

$$\Delta^2 + 4\Delta + 5 = 0.$$

quadratic formula.

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(15)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$\Delta = \frac{2(-2 \pm i)}{2}.$$

$$y_c = e^{4}(C_1 \cos t + C_2 \sin t).$$

For $y_p = ?$

$$yp = \frac{1}{D^3+D^2-7D-15} e^{xt}$$

$$= 1/64+16-28-15 \; e^{xt}$$

$$= \frac{1}{80-43} e^{xt}$$

Hence $y = y + yp.$

$$y = (C_1 \cos t + C_2 \sin t) + \frac{1}{37} e^{xt}$$

again put f $\ln x$ & $x = \cancel{\;}\ \ln x$

$$y = e^{34}(C_1 \cos i n x + C_2 \sin b g x) + \frac{1}{37} e^{4y} \text{ Ans}$$

Q3 $x^2 y''' + 2xy' - by = 10x^2$

Solution:

$y(1) = 1$ & $y'(1) = -6$

$$x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 6y = 10x^2$$

$$\Rightarrow \left(\frac{x^2 d^2}{dx^2} + \frac{2xd}{dx} - 6 \right) y = 10x^2$$

put $xD = \Delta \Rightarrow x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$x = e^t$ & $\log x = t$

$(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}$

$(\Delta^2 + \Delta - 6)y = 10e^{2t}$

The characteristic equation

$\Delta^2 + \Delta - 6 = 0$

$\Delta^2 + 3\Delta - 2\Delta - 6 = 0$

$\Rightarrow D(D+3) - 2(D+3) = 0$

$\Rightarrow (D+3)(D-2) = 0$

$D+3 = 0,\ \Delta - 2 = 0$

$\Delta = 2,\quad \Delta = -3.$

Since roots are real and distinct

For $y_c = ?$

$$y_c = C_1 e^{-3t} + C_2 e^{2t}$$

For $y_P = ?$

$$y_P = \frac{1}{\Delta^2 - \Delta - 6} \cdot 10^{2t}$$

$$= \frac{10}{D^2 - D - 6} e^{2t}$$

$$= 10 \ 1/0 \ e^{2t} \text{ fails.}$$

Now

$$10 \ 1/ d/dD (\Delta^2 + \Delta - 6) e^{2t}.$$

$$\Rightarrow 10 \frac{t}{2\Delta + 1} e^{2t}$$

$$= 10 \frac{1 \cdot t}{4 + 1} e^{2t}$$

$$y_P = 2te^{2t}$$

General Solution

$$y = y_c + y_P.$$

$= C_1 e^{-3t} + C_2 e^{2t} + 2te^{2t}$

$y = C_1 x^{-3} + C_2 x^2 + 2(\log x) x^2$ — (B)

put $y(1) = 1$ i.e $x = 1$, $y = 1$ in (B).

$1 = C_1 (1)^{-3} + C_2 (1)^2 + 2 \log(1)$.

$1 = C_1 + C_2$ — (C)

Now differentiate equ (B) w.r.t x.

$y_1 = -3C_1 x^{-4} + 2C_2 x + \frac{2}{x}(x^2) + 4x \log x$

Now put $y'(1) = -6$ i.e $y' = -6$ & $x = -6$

$-6 = -3C_1 + 2C_2 + 2 + 0$

$-6 = -3C_1 + 2C_2 + 2$

$-6 - 2 = -3C_1 + 2C_2 + 2$

$-8 = -3C_1 + 2C_2$ — (D).

Xing eq (ii) with (2) & ÷ ing from D

$$\begin{array}{r} 2C_1 + 2\cancel{C_2} = 2 \\ \mp 3C_1 \pm \cancel{2C_2} = \mp 8 \\ \hline 5C_1 = 10 \end{array}$$

$C_1 = 10/5$

$\boxed{C_1 = 2}$

$$-8 = -3(2) + 2c_2$$

$$-8 = -6 + 2c_2$$

$$2c_2 = -8 + 6$$

$$2c_2 = -2$$

$$c_2 = -\frac{2}{2}$$

$$\boxed{c_2 = -1}$$

Now put the vale of c_1 & c_2 in equ (B).

$$y = 2x^{-3} - x^2 + 2 \ln x \, x(x^2)$$

$$y = 2/x^3 - x^2 + 2x^2 \log x \text{ Ans.}$$

Q4 $x^2 y'' + 7xy' + 5y = x^5$

$y(x) = 2$ & $y'(1) = 2$

Solution:-

$$\frac{x^2 dy^2}{dx^2} + \frac{7x\, dy}{dx} + 5y = x^5$$

$$\Rightarrow \left(x^2 d^2/dx^2 + \frac{7x\, d}{dx} + 1\right) y = x^5 \quad — (A)$$

put $xD = D \Rightarrow x^2 D^2 = D(D-1) = D^2 - \Delta$

$x = e^t \Rightarrow \log x = t$ in eq (A)

$(D^2 - D + 7D + 5) y = e^{5t}$

$(D^2 + 6D + 5) y = e^{5t}$

By quadratic formula.

$$D = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$$

$$D = \frac{-6 \pm \sqrt{36-20}}{2}$$

$$= \frac{-6 \pm \sqrt{16}}{2}$$

$$= \frac{-6 \pm \sqrt{4^2}}{2}$$

$$= \frac{\cancel{2}(-3 \pm 2)}{\cancel{2}}$$

$$D = -3 \pm 2$$

Since roots are real and distinct

$$y_c = C_1 e^{-5t} + C_2 e^{-t}$$

For $y_P = ?$

$$y_P = \frac{1}{\Delta^2 + 6\Delta + 5} e^{5t}$$

$$= \frac{1}{(5)^2 + 6(5) + 5} e^{5t}$$

$$= 1/60 \; e^{5t}.$$

Now General solution is

$y = y_c + y_p$

$y = C_1 e^{-5t} + C_2 e^{-t} + 1/60\, e^{5t}$

$y = C_1 x^{-5} + C_2 x^{-1} + 1/60\, x^5$ — (B)

$x = 0$ put in this equation

$e^0 = 1$

put $y(0) = 2$ i.e $y = 2$ & $x = 2$

$2 = C_1 (2)^{-5} + C_2 (2)^{-1} + 1/60\, (2)^5$

$2 = -32 C_1 - 2 C_2 + 1/\cancel{60}_{15} \, (\cancel{32}^{8})$

$2 = -32 C_1 - 2 C_2 + 8/15$

$2 - \dfrac{8}{15} = -32 C_1 - 2 C_2$

$\dfrac{22}{15} = -32 C_1 - 2 C_2$ — (C)

Now differentiate w.r.t (x)

$y' = -5 C_1 x^{-6} - C_2 x^{-2} + 1/12\, x^4 \longrightarrow$

put $y'(1) = 2$ i.e $y' = 2$ & $x = 2$ in above eq

$$2 = -5c_1 c_2 - 6 - c_2 (2)^2 + 1/12 (2)^4$$

$$2 = -5c_1(-64) - c_2(4) + 1/12 (16)$$

$$2 = 320c_1 + 4c_2 + 4/3.$$

$$\Rightarrow 2 - \frac{4}{3} = 320c_1 + 4c_2$$

$$\Rightarrow 2/3 = 320c_1 + 4c_2 \quad \text{— (D)}.$$

Ling equ (c) with 2 & then
-ing equ from (D)

$$\frac{-44}{15} = 64c_1 + 4c_2.$$

$$-\frac{44}{15} = 64c_1 + 4c_2.$$

$$+2/3 = \pm 320c_1 \pm 4c_2$$

$$+$$

$$\frac{34}{15} = -256c_1$$

$$c_1 = \frac{34}{15} \times 256$$

$$\boxed{c_1 = 580}$$

Put the vale of C_1 in equ (C)

$$\frac{22}{15} = -32(580) - 2C_2$$

$$\frac{22}{15} = 18560 - 2C_2$$

$$\Rightarrow \frac{22}{15} + 18560 = -2C_2$$

$$\Rightarrow \frac{18561}{-2} = C_2.$$

$$\boxed{-9280 = C_2}$$

Now put the vale of C_1 & C_2 in equ (B)

$$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x^5$$

$$y = \frac{580}{x^5} - \frac{9280}{x} + 1/60\, x^5$$ ANS.

Q5 $(x+1)^2 y'' - 3(x+1)' y' + 4y = x^2$

Solution:

$$(x+1)^2 \frac{dy}{dx^2} - 3(x+1)\, dy/dx + 4y = x^2$$

$$(x+1)^2 \frac{d^2}{dx^2} - 3(x+1)\, d/dx + 4)\, y = x^2$$

$$(x+1)^2 \Delta^2 - 3(x+1)\Delta + 4]\, y = x^2 \text{ — } (A)$$

put $(x+1)\Delta = 0 \implies (x+1)^2 \Delta^2 = D(D-1) = \Delta^2 - D$

$x = e^t$ in eq (A)

$$[D^2 - D - 3D + 4]\, y = e^{2t}$$

$$(D^2 - 4D + 4)\, y = e^{2t}$$

$$(D^2 - 4D + 4)\, 2 = e^{2t}$$

for y_c we find the root

$$D^2 - 4D + 4 = 0$$

$$D^2 - 2D - 2D + 4 = 0$$

$$D - 2 - 2(D-2) = 0$$

$D - 2 = 0, \quad D = 2$

$D - 2 = 0 \quad D = 2$

So the roots are real and repeat

the general solution:

$$y = (C_1 + C_2 x)^{24}$$

$$y = (C_1 + C_3 x)^{23}$$

for $yp = ?$

$$yp = \frac{1}{D^2 - 4D + 4}$$

$$yp = \frac{2}{2D - 4} e^{2t}$$

if we put 2

$2D - 4 \Rightarrow 2(2) - 4 = 0$

we take again derivative

$$yp = 2/2 \, e^{-2t}$$

$$y = (C_1 + C_2 x)^{2t} + e^{2t} A$$