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PAPER :- Differential Equation

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Q1-

(i) $w = \sin(x+ct) + \cos(2x+2ct)$

Sol:→

Given $\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \rightarrow ①$

Now

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$= \frac{\partial}{\partial t}(\sin(x+ct)) + \frac{\partial}{\partial t}(\cos(2x+2ct))$$

$$\frac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct)$$

Now

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[c\cos(x+ct) - 2c\sin(2x+2ct)\right]$$

$$\boxed{\frac{\partial^2 w}{\partial t^2} = -c^2\sin(x+ct) - 4c^2\cos(2x+2ct)}$$

Now

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - 2\sin(2x+2ct)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct) - 2\sin(2x+2ct)\right]$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

Pg (2)

(1) $\Rightarrow$ $-c\sin(x+ct) - 4c^2\cos(2x+2ct) = c^2$

$[-\sin(x+2ct) - 4\cos(2x+2ct)$

$-c\sin\cancel{(x+ct)} - 4c^2\cos\cancel{(2x+2ct)} = -c\sin\cancel{(x+ct)} - 4c^2\cos(2x+2ct)$

$0=0$ (Satisfied)

(ii) $w = \tan(2x+ct)$

Now

$$\frac{\partial w}{\partial t} = c\sec^2(2x+ct)$$

&

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}(c\sec^2(2x+xt)$$

$$= c^2 \cdot 2\sec^2(2x+ct)\tan(2x+ct)$$

Now

$$\frac{\partial w}{\partial x} = 2\sec^2(2x+ct)$$

$$\frac{\partial^2 w}{\partial x^2} = 4\sec^2(2x+ct)\tan(2x+ct)$$

(1) $\Rightarrow$ $4c^2\sec^2\cancel{(2x+ct)}\tan(2x+ct) = 4c^2\sec^2\cancel{(2x+ct)}\tan(2x+ct)$

$0=0$ satisfied

Q2

$$f(x) = x, \quad -\pi < x \le 0$$
$$= 2x, \quad 0 \le x \le \pi$$

Sol:—

Given function is

$$f(x) = \begin{cases} x; & -\pi < x \le 0 \\ 2x; & 0 \le x \le \pi \end{cases}$$

We have to find the fourier co-efficient, a_0, a_n & b_n

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left[0 - \frac{\pi^{\cancel{2}}}{2}\right] + \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^{\cancel{2}}}{2} - 0\right]$$

$$\boxed{a_0 = \frac{-\pi}{2} + \pi = \frac{\pi}{2}} \longrightarrow (1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n+2(-1)^n-2}{n^2}\right] = \frac{(-1)^n-1}{\pi n^2}$$

So

$$\boxed{a_n = \begin{cases} \frac{-2}{\pi n^2} ; & \text{if } n \text{ is odd} \\ & \text{if } n \text{ is even} \end{cases}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos x}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

③

$$\boxed{b_n = \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] = \frac{-3\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}}$$

So the required fourier series is :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\left\{\frac{\sin nx}{n}\right\}$$

Pg 5

Q3. $y'' - 4y' + 13y = 8\sin 3x,\ y(0) = 1$ & $y'(0) = 2$

Sol:-

$y'' - 4y' + 13y = 8\sin 3x$

we have to find $y = y_c + y_p$

for y_c the characteristic (auxiliary) eqn is

$m^2 - 4m + 13 = 0$

$\Rightarrow m = \dfrac{4 \pm \sqrt{16-52}}{2} \Rightarrow m = \dfrac{4 + 6i}{2}$

$\Rightarrow m = 2 + 3i\ ;\ \alpha = 2\ \&\ \beta = 3$

So $y_c = e^{2x}\{c_1 \cos 3x + c_2 \sin 3x\}$

for y_p let

$y_p = \text{Imag}\left(\dfrac{1}{m^2 - 4m + 13}\, 8e^{3ix}\right)$

$= 8\,\text{Imag}\ \dfrac{e^{is}}{(3i)^2 - 4(3i) + 13}$

$= 8\,\text{Imag}\ \dfrac{e^{3is}}{-9 - 12i + 13}$

$= 8\,\text{Imag}\ \dfrac{e^{3ix}}{4 - 12i}$

$y_p = 2\,\text{Imag}\ \dfrac{e^{3is}}{(1-3i)} \times \dfrac{(1+3i)}{(1+3i)}$

$y_p = 2\,\text{Imag}\ \dfrac{(1+3i)(e^{3ix})}{(1)^2 - (3i)^2}$

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$$y_p = 2\, \text{Img}\, \frac{(1+3i)(e^{3ix})}{10}$$

$$y_p = \frac{2}{10}\left(\text{Imag}\,(1+3i)(\cos 3x + i \sin 3x)\right)$$

$$y_p = \frac{2}{10}\left(\sin 3x + 3\cos 3x\right)$$

So the general solution is

$$y = y_c + y_p$$

$$y = c_1 e^{2x}\cos 3x + c_2 e^{2x}\sin 3x + \frac{2}{10}\left(\sin 3x + 3\cos 3x\right)$$

Now use the initial condition $y(0) = 1$

$$y(0) = c_1 e^{0}\cos(0) + c_2 e^{(0)}\sin(0) + \frac{2}{10}\left(\sin(0) + 3x\cos\right.$$

$$1 = c_1(1) + 0 + 0 + \frac{2}{10}\left(3(1)\right)$$

$$1 = c_1 + \frac{6}{10} \rightarrow \boxed{c_1 = 1 - \frac{6}{10} = \frac{4}{10} = \frac{2}{5}}$$

Again use the another Initial condition

$$y'(0) = 2$$

So
$$y' = c_1 2e^{2x}\cos 3x + c_1 e^{2x}(-3\sin 3x) + c_2 2e^{2x}\sin 3x + c_2 e^{2x}(3\cos 3x) + \frac{2}{10}\left(\cos 3x - 3\sin 3x\right)$$

$$y'(0) = c_1 2e^{0}\cos(0) + c_1 e^{(0)}(-3\sin(0)) + c_2 2e^{(0)}\sin(0) + c_2 e^{(0)}(3\cos(0)$$

$$+\frac{2}{10}\left(\cos(0)-3\sin(0)\right)$$

$$2=2c_1+0+0+c_2\,3(1)+\frac{2}{10}\left(1-3(0)\right)$$

$$2=2c_1+3c_2+\frac{2}{10}$$

$$2=2(2/5)+3c_2+2/10 \qquad \boxed{\text{use } c_1=2/5}$$

$$1/3\left(2-4/5-2/10\right)=c_2 \Rightarrow \boxed{c_2=1/3\left(\frac{20-8-2}{10}\right)=1/3}$$

So the General solution is

$$y=2/5\,e^{2x}\cos 3x+1/3\,e^{2x}\sin 3x+\frac{2}{10}\left[\sin 3x+3\cos 3x\right]$$

is the required solution.

Q4

$(D^2 - DD')z = \cos x \cos 2y$

Sol:-

The auxiliary equation is

$m^2 - m = 0 \Rightarrow m = 0, m = 1$

Hence the complementry function is given by

$Z_c = f_1(y) + f_2(y+x)$

for the particular Integral, we have

$$Z_P = \frac{1}{D^2 - DD'}\left[\cos(x-2y) + \cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\cos(x+2y) + \frac{1}{D^2 - DD'}\cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{-1+2}\cos(x+2y) + \frac{1}{-1-2}\cos(x-2y)\right]$$

$$= \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Hence the complete solution is given by

$$Z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$