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Q2 Evaluate

$$\int \frac{4x^3+10x+4}{2x^2+x}$$

Solution:

Split the integral into multiple integrals

$$\Rightarrow \int 2x\,dx + \int -1\,dx + \int \frac{11x+4}{2x^2+x}\,dx$$

Since 2 is constant with respect to x, move 2 out of the integral

$$\Rightarrow 2\int x\,dx + \int -1\,dx + \int \frac{11x+4}{2x^2+x}\,dx$$

By the power Rule, the integral of x with respect to x is $\frac{1}{2}x^2$

$2\left(\frac{1}{2}x^2+C\right) =$

$$\Rightarrow 2\left(\frac{1}{2}x^2+C\right)+\int -1\,dx+\int \frac{11x+4}{2x^2+x}\,dx$$

Since -1 is constant with respect to x move -1 out of the integral

$$\Rightarrow 2\left(\frac{1}{2}x^2+c\right)-x+C+\int \frac{11x+4}{2x^2+x}\,dx$$

Simplify by factoring out.

$$\Rightarrow 2\left(\frac{x^2}{2}+C\right)-x+C+\int \frac{11x+4}{x(2x+1)}\,dx$$

write the fraction using partial fraction decomposition

$$\Rightarrow 2\left(\frac{x^2}{2}+c\right)-x+C+\int \frac{A_1}{x}+\frac{A_2}{2x+1}\,dx$$

Replace each of the partial fraction coefficients in $\frac{A}{x}+\frac{B}{2x+1}$ with the values found A & B

$$\Rightarrow 2\left(\frac{x^2}{2}+C\right)-x+C+\int \frac{4}{x}+\frac{3}{2x+1}\,dx$$

Split the single integral into multiple integrals

$$\Rightarrow 2\left(\frac{x^2}{2}+c\right)-x+C+\int\frac{4}{x}\,dx+\int\frac{3}{2x+1}\,dx$$

Since 4 is constant with respect to x move 4 out of the integral

$$\Rightarrow 2\left(\frac{x^2}{2}+c\right)-x+C+4\int\frac{1}{x}\,dx+\int\frac{3}{2x+1}\,dx$$

The integral of $\frac{1}{x}$ with respect to x is $\ln(|x|)$

$$\Rightarrow 2\left(\frac{x^2}{2}+c\right)-x+C+4\left(\ln(|x|)+C+\int\frac{3}{2x+1}\,dx\right.$$

Since 3 is constant with respect to x move 3 out of the integral

$$\Rightarrow 2\left(\frac{x^2}{2}+c\right)-x+C+4(\ln|x|+c)+3\int\frac{1}{2x+1}\,dx$$

Let $u=2x+1$. Then $du=2dx$, so $\frac{1}{2}du=dx$. Rewrite using u & du

$$\Rightarrow 2\left(\frac{x^2}{2}+c\right)-x+C+4\left(\ln(|x|)+C+3\int\frac{1}{u}\cdot\frac{1}{2}\,du\right.$$

Simplify

$$\Rightarrow 2\left(\frac{x^2}{2}+c\right)-x+c+4\left(\ln(|x|)+c+3\int\frac{1}{2u}\,du\right.$$

Since $\frac{1}{2}$ is constant with respect to u,

move $\frac{1}{2}$ out of the integral

$$\Rightarrow 2\left(\frac{x^2}{2}+c\right)-x+c+4\left(\ln(|x|)+c\right)+3\left(\frac{1}{2}\int\frac{1}{u}\,du\right)$$

combine $\frac{1}{2}$ & 3

$$\Rightarrow 2\left(\frac{x^2}{2}+c\right)-x+c+4\left(\ln(|x|)+c+\frac{3}{2}\int\frac{1}{u}\,du\right.$$

The integral of $\frac{1}{u}$ with respect to

u is $\ln(|u|)$

$$\Rightarrow 2\left(\frac{x^2}{2}+c\right)-x+c+4\left(\ln(|x|)+c\right)+$$

$$\frac{3}{2}\ln(|u|)+c$$

Replace all occurrences of u with $2x+1$

$$\Rightarrow x^2-x+4\ln(|x|)+\frac{3}{2}\ln(|2x+1|)+C$$

Ans

Q 3 a $\int_0^2 x^2 e^x \, dx$

Solution

Integration by parts

$\int uv' = uv - \int u'v$

$u = x^2$ $\quad v' = e^x$

$u' = 2x$ $\quad v = e^x$

$= x^2 e^x - \int 2xe^x \, dx$

Now solving xe^x

Integration by parts

$u = x$ $\quad v' = e^x$

$u' = 1$ $\quad v = e^x$

$= x^2 e^x - 2\left[xe^x - \int e^x \, dx \right]$

$= x^2 e^x - 2\left[xe^x - e^x \right] \Big|_0^2$

$= x^2 e^x - 2xe^x + 2e^x \Big|_0^2$

$= \left((2)^2 e^2 - 2(2) e^2 + 2e^2 \right) - 2e^0$

$= \cancel{4e^2} - \cancel{4e^2} + 2e^2 - 2$

$= 2e^2 - 2$

Q 3 b $\displaystyle\int_1^2 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Solution

$u = \sqrt{x} \qquad \dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$

so $dx = 2\sqrt{x}\,du$

$= \displaystyle\int_1^2 2\sin u \; du$

$= 2\displaystyle\int_1^2 \sin u \; du$

$= -2\cos u \Big|_1^2$

$u = \sqrt{x}$

$= -2\cos\sqrt{x} \Big|_1^2$

$= \left[-2\cos\sqrt{2}\right] - \left[-2\cos\sqrt{1}\right]$

$= -2\cos\sqrt{2} + \sqrt{2}\cos\sqrt{1}$

$= 2\left[\cos\sqrt{1} - \cos\sqrt{2}\right]$

Q 1 Find where PQ where P is the point in three-dimesional space with coordinates (4, 1, 3) & the point Q with coordinates (1, 2, 4). Find the distance b/w P & Q. Further, find the position vector of the point dividing PQ in the ratio 1:3.

Solution

Coordinate of P = (4, 1, 3)

$OP = 4i + 1j + 3k$

or $OQ = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (i + 2j + 4k) - (4i + 1j + 3k)$

$= -3i + 1j + 3k \rightarrow (1)$

now distance between P & Q = |PQ|

$= \sqrt{(-3)^2 + (1)^2 + (1)^2}$

$= \sqrt{9+1+1}$

$= \sqrt{11} \rightarrow (2)$

Let M be the point when devided PQ in ratio 1:3, then by ratio theorem position vector of $M = \overrightarrow{OM}$

$$= \frac{3(4i + 1j + 3k) + (1)(1 + 2j + 4k)}{1+3}$$

$$= \frac{12i + 3j + 9k + i + 2j + 4k}{4}$$

$$= \frac{13i + 5j + 13k}{4} \rightarrow (3)$$

Hence eq (1), (2), (3) are the required solution.

Q4 $U(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$

Solution:

Laplace eq

$U_{xx} + U_{yy} + U_{zz} = 0$

first find U_{xx}

$$U_x = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}}\right)$$

using quotien rule

$$U_x = \frac{\frac{d}{dx}(1) \cdot \sqrt{x^2 + y^2 + z^2} - \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2}}{\left(\sqrt{x^2 + y^2 + z^2}\right)^2}$$

$$U_x = \frac{-\frac{\partial}{\partial x} \sqrt{x^2+y^2+z^2}}{\left(\sqrt{x^2+y^2+z^2}\right)^2}$$

$$U_x = \frac{-\frac{\partial}{\partial x} (x^2+y^2+z^2)}{x^2+y^2+z^2}$$

$$-\left(\frac{1}{2}-1\right)$$

$$x^2+y^2+z^2 = 2 = \sqrt{x^2+y^2+z^2}$$

$$U_x = \frac{-\cancel{2}x}{x^2+y^2+z^2\ \cancel{2}\cdot\sqrt{x^2+y^2+z^2}}$$

$$U_x = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$U_{xx} = \frac{\partial}{\partial x}\left[\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right]$$

use quotient rule

$$U_{xx} = \frac{-(x^2+y^2+z^2)^{3/2} - \frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{3}{2}} \cdot x}{(x^2+y^2+z^2)^3}$$

$$U_{xx} = \frac{-(x^2+y^2+z^2)^{3/2} - \frac{3}{2}(x^2+y^2+z^2)^{\frac{3}{2}-1} \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2)x}{(x^2+y^2+z^2)^3}$$

$$U_{xx} = \frac{-(x^2+y^2+z^2)^{3/2} - \frac{3}{\cancel{2}} \cdot x^2+y^2+z^2\,(-\cancel{2}x^2)}{(x^2+y^2+z^2)^3}$$

so $$U_{xx} = \frac{-(x^2+y^2+z^2)^{3/2} - 3\sqrt{x^2+y^2+z^2}\cdot x^2}{(x^2+y^2+z^2)^3}$$

Now solving for

$U_{yy} =$

first $$U_x = \frac{\partial}{\partial y}\left[\frac{1}{x^2+y^2+z^2}\right]$$

using quotient rule

$$U_y = \frac{-\partial/\partial y\,(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}^{\,2}}$$

$$U_y = \frac{\frac{\partial}{\partial y}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{-(\frac{1}{2}-1)} \cdot 2\sqrt{(x^2+y^2+z^2)}^{\,2}}$$

$$U_y = \frac{-y}{(x^2+y^2+z^2)^{3/2}}$$

Now U_{yy}

$$U_{yy} = \frac{\partial}{\partial y}\left(\frac{-y}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$U_{yy} = \frac{(x^2+y^2+z^2)^{3/2} - \partial/\partial y\,(x^2+y^2+z^2)^{3/2} \cdot y}{(x^2+y^2+z^2)^3}$$

$$U_{yy} = \frac{-(x^2+y^2+z^2)^{3/2} - \frac{3}{\cancel{2}}\sqrt{x^2+y^2+z^2}\,(\cancel{2}y^2)}{(x^2+y^2+z^2)^3}$$

$$U_{yy} = \frac{-(x^2+y^2+z^2)^{3/2}\quad 3\sqrt{x^2+y^2+z^2}\cdot y}{x^2+y^2+z^2}$$

Now U_{zz}

$$U_z = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

$$U_{zz} = \frac{-(x^2+y^2+z^2)^{3/2} - 3\sqrt{x^2+y^2+z^2}\cdot z^2}{(x^2+y^2+z^2)^3}$$

$U_{xx} + U_{xx} + U_{zz} = 0$

if it is satisfid then it is laplace equation

$$= \frac{(x^2+y^2+z^2)^{3/2} - 3\sqrt{x^2+y^2+z^2}\,(x^2)}{(x^2+y^2+z^2)^3} + \left(\frac{-x^2+y^2+z^2)^{3/2} - 3\,x^2+y^2+z^2(y)^2}{x^2+y^2+z^2}\right) +$$

$$- \frac{(x^2+y^2+z^2)^3 - 3\sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)^3}$$

$$= \frac{-(x^2+y^2+z^2)^{2/3} + 3\,x^2+y^2+z^2(x^2) - (x^2+y^2+z^2)^{3/2} + 3\sqrt{x^2+y^2+z^2}(y)^2 - (x^2+y^2+z^2)^{3/2} + 3\sqrt{x^2+y^2+z^2}\,(z)^2}{(x^2+y^2+z^2)^3}$$

$$= \frac{-3(x^2+y^2+z^2)^{3/2} + 3\sqrt{x^2+y^2+z^2}\;(x^2+y^2+z^2)}{(x^2+y^2+z^2)^3}$$

$$= \frac{-3(x^2+y^2+z^2)^{3/2} + 3(x^2+y^2+z^2)^{1/2}\cdot(x^2+y^2+z^2)}{(x^2+y^2+z^2)^3}$$

$$= \frac{-3(x^2+y^2+z^2) + 3(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)}$$

$$= 0$$

$U_{xx} + U_{yy} + U_{zz} = 0$ Hence proved

it is laplace equation