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Subject = Differential Equations

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Question No 1

Find the fourier series representation of
 $f(t) = 1 + t, -\pi \leq t \leq \pi$

Solution:-

Here we use the formula

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos t + \sum_{n=1}^{\infty} b_n \sin t \quad \text{--- (1)}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + t) dt$$

$$a_0 = \frac{1}{2\pi} \left[t + \frac{t^2}{2} \right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{2\pi} \left(\pi - (-\pi) \right) + \frac{\pi^2}{2} - \left(\frac{-\pi^2}{2} \right)$$

$$a_0 = \frac{1}{2\pi} \left(2\pi + \frac{2\pi^2}{2} \right)$$

$$a_0 = \frac{1}{2\pi} \left(2\pi + \pi^2 \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1+t) (\cos nt) dt$$

$$a_n = \frac{1}{\pi} \left((1+t) \frac{\sin nt}{n} \Big|_{-\pi}^{\pi} - \int \left(\frac{\sin nt}{n} \frac{d}{dt} (1+t) \right) dt \right)$$

$$a_n = \frac{1}{\pi} \left((1+t) \frac{\sin nt}{n} \Big|_{-\pi}^{\pi} - \frac{\cos nt}{n^2} \Big|_{-\pi}^{\pi} \right)$$

$$a_n = \frac{-1}{n^2\pi} \left(\cos n\pi - \cos n(-\pi) \right)$$

$$a_n = \frac{-1}{n^2\pi} \left(-1 - (-1) \right)$$

$$a_n = 0$$

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$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1+t) \sin nt \, dt$$

$$b_n = \frac{1}{\pi} \left((1+t) \int_{-\pi}^{\pi} \sin nt - \int_{-\pi}^{\pi} \left(\sin \frac{d}{dt} (1+t) \right) dt \right)$$

$$b_n = \frac{1}{\pi} \left(\frac{(1+t)(-\cos nt)}{n} \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \left(-\frac{\cos nt}{n} \right) dt \right)$$

$$b_n = \frac{1}{\pi} \left(\frac{-(1+t)(\cos nt)}{n} \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \left(\frac{\sin nt}{n^2} \right) dt \right)$$

$$b_n = \frac{-1}{n\pi} \left((1+\pi)(\cos n\pi) - (1+(-\pi))(\cos n(-\pi)) \right)$$

$$b_n = \frac{-1}{n\pi} \left(\cancel{\cos n\pi} + \pi \cos n\pi - \cancel{\cos n\pi} + \pi \cos n\pi \right)$$

$$b_n = \frac{-1}{n\pi} (2\pi \cos n\pi)$$

Here $\cos n\pi = \frac{(-1)^{n+1}}{n}$

$$b_n = \frac{2}{n} (-1)^{n+1}$$

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so the equation become

$$f(x) = \frac{1}{2\pi} (2\pi + \pi^2) + 0 + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

Question No 2

calculate the characteristic equation the Eigen values of the system - where A is given by

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix}$$

Solution:-

Step = 01

we have

$$(A - \lambda I)x = 0$$

A = Given Matrix

Step : 02

we have the characteristic equation is given by:

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 3 & 1-\lambda & 4 \\ 0 & 2 & 2-\lambda \end{vmatrix} = 0$$

Step: 03

$$\lambda^3 - \left| \begin{smallmatrix} \text{Sum of} \\ \text{Diagonal elem} \end{smallmatrix} \right| \lambda^2 + \left| \begin{smallmatrix} \text{sum of} \\ \text{Diagonal minors} \end{smallmatrix} \right| \lambda - |A| = 0 \quad (B)$$

$$\text{Sum of Diagonal elements} = 1+1+2 = 4$$

$$\begin{aligned} \text{Sum of Diagonal minors} &= \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \\ &= (-6) + (2) + (1) \\ &= (-6) + (2) + (1) \\ &= -6 + 2 + 1 \\ &= -3 \end{aligned}$$

By putting values in eq (B):

$$\lambda^3 - 4\lambda^2 - 3\lambda - |A| = 0 \rightarrow (C)$$

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1 \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} - 0 \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} \\ &= 1(2-8) - 0 + 1(6-0) \\ &= -6 + 6 \\ &= 0 \end{aligned}$$

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By putting values in (6) =

$$\lambda^3 - 4\lambda^2 - 3\lambda - 0 = 0$$

$$\lambda^3 - 4\lambda^2 - 3\lambda = 0$$

$$\lambda(\lambda^2 - 4\lambda - 3) = 0$$

$$\lambda = 0$$

$$\lambda^2 - 4\lambda - 3 = 0$$

Using quadratic formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1$$

$$b = -4$$

$$c = -3$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 + 12}}{2} = \frac{4 \pm \sqrt{28}}{2}$$

$$\lambda = \frac{4 + \sqrt{28}}{2}, \quad \lambda = \frac{4 - \sqrt{28}}{2}$$

we have eigenvalues.

$$\lambda = \left(0, \frac{4 + \sqrt{28}}{2}, \frac{4 - \sqrt{28}}{2} \right)$$

required
Solution.

Question 3

Solve the following system of linear equations

$$5x + 0 + 4z + 2m = 3$$

$$x - y + 2z + m = 1$$

$$4x + y + 2z + 0 = 1$$

$$x + y + z + m = 0$$

Solution:

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_1 \\ R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 0 & 2 & -1 & 0 & -1 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -1 & +6/5 & +4/5 & 3/5 \\ 0 & 2 & -1 & 0 & -1 \end{array} \right] \quad \underbrace{-1/5 \times R_3}$$

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -1 & 6/5 & 4/5 & 3/5 \\ 0 & 0 & 7/5 & 8/5 & 1/5 \end{array} \right] \quad \underbrace{5 \times R_3 \text{ and } 5 \times R_4}$$

$$\left[\begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \quad \underbrace{5R_3 \text{ and } 5R_4}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \quad \underbrace{1/5 \times R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -1 & 6/5 & 1/5 & 2/5 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \quad \underbrace{R_2 \times 5}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \quad R_3 - R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 1 & 8/7 & 1/7 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right] \quad \begin{array}{l} R_3 \leftrightarrow R_4 \\ 1/7 \times R_3 \\ 1/3 \times R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 1 & 1 & -4/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right] \quad R_2 \times -5$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 6 & 1 & 2 \\ 0 & 0 & 1 & 1 & -4/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 0 & -5 & 26/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1/2 & 3/4 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right] \quad S/4 \times R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 126/84 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 0 & 0 & -11/21 \\ 0 & 0 & 0 & 0 & 1/3 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3/4 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

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$$(x, y, z, m) = \left(\frac{3}{4}, \frac{31}{21}, \frac{-11}{21}, \frac{1}{3} \right)$$

$$x = \frac{3}{4}$$

$$y = \frac{31}{21}$$

$$z = \frac{-11}{21}$$

$$m = \frac{1}{3}$$

Question No 4

$$\underline{u(x, t) = \sin(x + \gamma t)}$$

solution:-

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x, t) = \sin(x + \gamma t) \quad \text{is}$$

solution of $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

it will satisfy the above equation

$$\frac{\partial u}{\partial t} = \cos(x + \gamma t) \cdot \frac{d}{dt}(x + \gamma t)$$

$$\frac{\partial u}{\partial t} = \gamma \cos(x + \gamma t)$$

again

$$\frac{\partial^2 u}{\partial t^2} = -\gamma \sin(x + \gamma t) \frac{\partial t}{\partial t}(x + \gamma t)$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = -4 \sin(x + \alpha t)} \rightarrow A$$

Now $\frac{\partial u}{\partial x} = \cos(x + \alpha t)$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x + \alpha t)$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} = -\sin(x + \alpha t)} \rightarrow B$$

Comparing A and B

$$c = 2$$

$$\Rightarrow -4 \sin(x + \alpha t) = -c^2 \sin(x + \alpha t)$$

$$\Rightarrow -4 \sin(x + \alpha t) + c^2 \sin(x + \alpha t) = 0$$

This is possible if $c = \pm 2$

$$-4 \sin(x + \alpha t) + (\pm 2)^2 \sin(x + \alpha t) = 0$$

$$\boxed{0 = 0}$$

This $u(x, t) = \sin(x + \alpha t)$