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Q No1:: Define the following terms.

(1) An ~~Org~~ orthogonal Matrix:

Ans: Orthogonal Matrix:-
= In linear algebra, an orthogonal matrix is a square matrix whose columns and rows are orthogonal unit vectors (orthogonal vectors).
One way to express this is

$$Q^t Q = Q Q^t = I$$

where Q^t is the transpose of Q and I is the identity matrix.
This leads to the equivalent characterization.
a matrix Q is orthogonal if its transpose is equal to its inverse.

$Q^t = Q^{-1}$

where Q^{-1} is the inverse of Q.

Example:

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $A^t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A \cdot A^t = I$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

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(2) **A basis for a vector space.**

In mathematics, a set B of elements (vectors) in a vector space V is called a basis, if every element of V may be written in a unique way as a (finite) linear combination of elements of B. The coefficients of this linear combination are refferred to as components or coordinates on B of the vector. The elements of basis are called basis vector.

P.T.O

Every vector space has a basis, that is a maximum linearly independent subset.

Example: $P = \{ P_2 = at^2 + bt + c$

$P_2 : \{t^2, t, 1\}$ natural basis for P.

$M_2 : \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$R^3 : \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$R_2 : \{ [1, 0], [0, 1] \}$

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(3) **The span of a set of vectors:-**

In linear algebra, the linear span (also called the linear hull or just span) of a set S of vectors in a vector space is the smallest linear ~~subset~~ subspace that contains the set. It can be characterized either as the intersection of all linear subspaces that contains S, or as the set of linear combinations of elements of S. The linear span of a set of vectors is therefore a vector space. Spans can be generalized to matroids and modules.

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4:. **The dimension of a vector space:.**

In mathematics, the dimension of a vector space V is the cardinality (i.e the number of vectors) of a basis of V over its base field. it is sometimes called Hamel dimension (after Georg Hamel) or algebraic dimension to distinguish it from other types of dimension.

For every vector space there exists a basis, and all bases of a vector space have equal cordinality, as a result the dimension of a vector space is uniquely defined. we say V is finite dimensional if the dimension of V is finite and infinite dimensional if its dimension is infinite.

Example:- The vector space R^3 has

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

as a standard basis, and therefore we have $\dim_R(R^3) = 3$. More generally, $\dim_R(R^n) = n$, and even more generally, $\dim_F(F^n) = n$ for any field F.

5. **An eigen vector:-** To understand the eigen vector first we know about eigen value.

Def: Let A be an $n \times n$ matrix. The real number λ is called an eigenvalue of A if there exists a nonzero vector x in R^n such that

$$Ax = \lambda x.$$

Now eigen vector: In linear algebra an eigenvector or characteristic vector of a linear transformation is a nonzero vector that changes at most by a scalar factor when that linear transformation is applied to it. The corresponding eigenvalue is the factor by which the eigenvector is scaled.

Geometrically, an eigenvector, corresponding to a real nonzero eigenvalue, points in a direction in which it is stretched by the transformation and the eigenvalue is the factor by which it is stretched.

Every nonzero vector x satisfying is called an eigenvector of A associated with the eigenvalue λ. We might mention that the word "eigenvalue" is a hybrid one. Eigenvalues are also called proper values.

6:- A subspace of vector Space:-

In mathematics and more specifically in linear algebra, a linear subspace also known as a vector subspace is a vector space that is a subset of some larger vector space. A linear subspace is usually called simply a subspace when the context serves to distinguish it from other types of subspaces.

If V is a vector space over a field K and if W is a subset of V, then W is a subspace of V if under the operations of V, W is a vector space over K. Equivalently a nonempty subset W is a subspace of V if, whenever w_1, w_2 are elements of W and α, β are elements of K, it follows that $\alpha w_1 + \beta w_2$ is in W.

Example:-

Let the field K be the set R of real numbers, and let the vector space V be the real coordinate space R^3. Take W to be the set of all vectors in V whose last component is 0. Then W is a ~~subset~~ subspace of V.

P.T.O

Proof:: (1) Given u and v in w, then they can be expressed as $u = (u_1, u_2, 0)$ and $v = (v_1, v_2, 0)$, then $u+v = (u_1+v_1, u_2+v_2, 0+0) = (u_1+v_1, u_2+v_2, 0)$

Thus u+v is an element of w, too.

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7:. The kernel of a linear transformation:-

Related to 1-1 linear transformations is the idea of the kernel of a linear transformation.

Definition:- The kernel of a linear transformation L is the set of all vectors v such that

$$L(v) = 0$$

Example:-
Let L be the linear transformation from $M^{2\times2}$ to P^1 defined by

$$L\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a+d)+(b+c)^r$$

Then fined the kernel of L is the set of all matrices of the form

$$\begin{bmatrix} a & b \\ -b & -a \end{bmatrix}$$

Notice that this set is a ~~subset~~ subspace of $M^{2\times2}$.

P.T.O

Theorem:- The kernel of a linear transformation from a vector space V to a vector space W is a subspace of V.

Proof:- Suppose that u and v are vectors in the kernel of L. Then:

$$L(u) = L(v) = 0$$

we have

$$L(u+v) = L(u) + (v) = 0+0 = 0$$

and

$$L(cu) = cL(u) = c0 = 0$$

Hence u+v and cu are in the kernel of L. We can conclude that the kernel of L is a subspace of V.

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8: The nullity of a linear transformation:

The nullity of a linear transformation of a vector spaces is the dimension of its null space. The nullity and the map rank add up to the dimension of a result sometimes known as the rank nullity theorem.

If $T : V^4 \to V^3$ is a linear transformation then nullity (T) + rank $(T) = 4$.

Proof:-

Transformation, then nullity (T) + rank $(T) = 4$

Nullity $(T) = \#$ non-leading columns in the row-echelon form of the matrix $A (3 \times 4)$.

rank $(T) = \#$ leading columns in the r.r.d A.

So nullity (T) + rank $(T) = \#$ columns of $A = 4$.

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9: The image of linear transformation:-

The image of a linear transformation or matrix is the span of the vectors of the linear transformation. (Think of its as what vectors you can get from applying the linear transformation or multiplying the matrix by a vector.) It can be written as Im(A).

For example:- consider the matrix (call it A):

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$$

multiplying this by a 2x1 gives a 3x1 matrix. However, regardless of what vector is chosen & multiply by, there are some vectors that can't be the result. Thus, these vectors are not in the image of A. The vectors that are possible belong to the span of A. In this case, the span can be represented by a "parametized"

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} s \\ 2t \\ t \end{bmatrix} = Im(A).$$

11 **The characteristic polynomial of a square matrix.**

Ans: In linear algebra the characteristic polynomial of a square matrix is a polynomial which is invariant under matrix similarity and has the eigenvalues as roots. It has the determinant and the trace of the matrix as coefficient. The characteristic polynomial of an endomorphism of vector space of finite dimension is the characteristic polynomial of the matrix of the endomorphyism over any base; it does not depend on the choice of a basis. The characteristic equation is the equation obtained by equating to zero the characteristic polynomial.

Formula: Consider an $n \times n$ matrix A. The characteristic polynomial of A, denoted by $P_A(t) = \det(tI - A)$

where I denotes the n-by-n identy matrix.

Some authors define the characteristic polynomial to be $\det(A - tI)$.

P.T.O

That polynomial differs from the one defined there by a sign $(-1)^n$, so it makes no difference for properties like having as roots the the eigenvalues of A. however the difinition above always gives a monic polynomial, whereas the alternative definition is monic only when n is even.

Example:

Suppose we want to compute the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

we now compute the determination of $tI - A = \begin{bmatrix} t-2 & -1 \\ 1 & t-0 \end{bmatrix}$.

which is $(t-2)t - 1(-1) = t^2 - 2t + 1$, the characteristic polynomial of A.

Another example hyperbolic function and hyperbolic angle angle φ. For matrix take

$$A = \begin{bmatrix} \cosh(\varphi) & \sinh(\varphi) \\ \sinh(\varphi) & \cosh(\varphi) \end{bmatrix}$$

Its characteristic polynomial is

$$\det(tI - A) = (t - \cosh(\varphi))^2 - \sinh^2.$$

12. An equelence relation:

Ans A Relation between elements of a set which is reflexive, symmetric, and transitive and which defines exclusive classes whose members bear the relation to each other and not to those in other classes.

In mathematics an equivalence relation is a binary relation that is reflexive, symmetrin and transitive. The relation (is equal to) is the canonical example of an equivalence relation where for any objects a, b and c:

$a = a$, if $a = b$, then $b = a$, and $b = c$ then $a = c$.

① $a = a$ (reflexive property).

② if $a = b$ then $b = a$ (symmetric property) and

③ if $a = b$ and $b = c$ then $a = c$ transitive property.

13 A homogeneous solution to a linear system of equations.

Ans: For a homogeneous system of equations $ax+by=0$ and $cx+dy=0$, the situation is slightly different. These lines pass through the origin. Thus, there is always at least one solution, the point $(0,0)$. If the slopes a/b and $-c/d$ are equal then there are an infinite number of solutions since the lines are identical. But as ~~well~~ we have seen the slopes of these lines are equal when the determinant of the coefficient matrix is zero.

Thus for homogeneous systems we have the following result.

A $n \times n$ homogeneous system of linear equations has a unique solution (the trivial solution) if and only if its determinant is non-zero. If this determinant is zero, then the system has an infinite number of solutions.

P.T.O

Example: The system of equations $2x+y=0$ and $x-y=0$ has exactly one solution since the slopes of the lines are different i.e. the determinant

$$\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = 2(-1)-1(1) = -3$$

is non-zero. The solution is of course $(0,0)$.

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146 **A particular solution to a linear system of equation:**

Ans:- In mathematics, a system of linear equations (or linear system) is a collection or more linear equations involving the same set of variables. [1][2][3][4][5] For example

$$3x + 2y - z = 1$$
$$2x - 2y + 4z = -2$$
$$-x + \frac{1}{2}y - z = 0$$

is a system of three equations

P.T.O

in the three variables x, y, z. A Solution to a linear system is an assignment of a values to the variables such that all the equations are simultaneously satisfied. A Solution to the system above is given by

$$x = 1$$
$$y = -2$$
$$z = -2$$

Since it makes all three equations valid. The word "system" indicates that the equations are to be considered collectively, rather than individually.

Examples:-

The system of one equation in one unknown $2x = 4$ has solution

$$x = 2.$$

However, a linear system is commonly considered as having at least two equations.

15 ## A General solution to a linear system of equation:

A General system of m linear equations with n unknowns can be written as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m,$$

where $x_1, x_2, \ldots, x_n$ are the unknowns, $a_{11}, a_{12}, \ldots, a_{mn}$ are the coefficients of the system and $b_1, b_2, \ldots b_m$ are the constant terms.

often the coefficients and unknowns are real or complex numbers, but integers and rational numbers are also seen, as are polynomials and elements of an abstract algebraic structure.

One extremely helpful view is that each unknown is a weight for a columns vector in a linear combination.

P.T.O

$$x_1\begin{bmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{bmatrix}+x_2\begin{bmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{bmatrix}+\cdots+x_n\begin{bmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_m\end{bmatrix}$$

This allows all the language and theory of vector spaces (or more generally, modules) to be brought to bear. For example the collection of all possible linear combinations of the vectors on the left-hand side is called their span, and the equations have a solution just when the right-hand vectors is within that span. if every vector within that span has exactly one expression as a linear combination of the given left-hand vectors, then any solution ~~is~~ is unique. In any event, the span has a basis of linearly independent vectors that do guarantee exactly one expression; and the number of vectors in that basis (its dimension) cannot be larger than m or n, but it can be smaller. This is important because if we have m independent vectors a solution is guaranteed m independent vectors a solution is guaranteed regardless of right-hand side, and otherwise not guaranteed.

16: The direct sum of a pair of subspaces of a vector space.

The direct sum is an operation from abstract algebra, a branch of mathematics. For example, the direct sum $R \oplus R$, where R is real coordinate space, is the cartesian plane, R^2.
To see how direct sum is used in abstract algebra, consider a more elementary structure in abstract algebra, the abelian group. The direct sum of two abelian groups A and B is another abelian groups $A \oplus B$ consisting of the ordered pairs (a, b) where $a \in A$ and $b \in B$. (confusingly this ordered pair is also called the cartesian product of the two groups). To add ordered pairs, we define the sum, $(a, b) + (c, d)$ to be $(a+c, b+d)$; in other words addition is defined coordinate wise. A similar process can be used to form the direct sum of any two algebraic structure. Such as rings, modules, and vector spaces.

P.T.O

we can also form direct sums with any finite number of summands for example $A \oplus B \oplus C$, provided A, B and C are the same kinds of algebraic structures (that is, all groups, rings, vector spaces etc). This relies on the fact that the direct sum is associative up to ~~isom~~ isomorphism. That is

$(A \oplus B) \oplus C \cong A \oplus (B \oplus C)$ for any algebraic structures A, B and C of the same kind. The direct sum is also commutative up any algebraic structures A and B of the same kind.

Example:

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
which is the same as vector addition

Given two structure A and B, their direct sum is written as $A \oplus B$. Given an indexed family of structures A_i, indexed with $i \in I$, the direct sum may be written $A = \oplus_{i \in I} A_i$. Each A_i is called a direct summand of A.

Q3:- Compute the following determinants.

(i) $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Sol:

$$|D| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$|D| = (1\times4) - (3\times2)$$

$$|D| = 4 - 6$$

$$|D| = -2$$

(ii) $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Sol:

let $D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$|D| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$|D| = \{1(45-48) - 4(18-24) + 7(12-15)\}$$

P.T.O

$$|D| = \{1(-3) - 4(-6) + 7(-3)\}$$

$$|D| = -3 + 24 - 21$$

$$|D| = 0$$

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(iii) $\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$

Sol: let $D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$

$$|D| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$$

$R_2 - 5R_1$
$R_3 - 9R_1$
$R_4 - 13R_1$

P.T.O

$$|D| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -20 & -24 \\ 0 & -12 & -24 & -46 \end{vmatrix}$$

$$|D| = \begin{vmatrix} -4 & -8 & -12 \\ -8 & -20 & -24 \\ -12 & -24 & -46 \end{vmatrix}$$

$$|D| = \{(-4(-20\times-46) - (-24\times-24)) + (8(-8\times-46) - (-12\times-24))$$
$$(-12(-8\times-24) - (-12)\times(20)))$$

$$|D| = -4(920-576) + 8(368-288) - 12(192-240)$$

$$|D| = -4(344) + 8(80) - 12(-48)$$

$$|D| = -1376 + 640 + 576$$

$$|D| = -160 \text{ Ans.}$$

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(iv) $\det \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}$

Sol.

Let $D = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{vmatrix}$

$R_2 - 6R_1;\ R_3 - 11R_1;\ R_4 - 16R_1;\ R_5 - 21R_1$

$$|D| = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -5 & -10 & -25 & -20 \\ 0 & -10 & -20 & -38 & -40 \\ 0 & -15 & -30 & -45 & -60 \\ 0 & -20 & -40 & -60 & -80 \end{vmatrix}$$

Now

$$|D| = \begin{vmatrix} -5 & -10 & -25 & -20 \\ -10 & -20 & -38 & -40 \\ -15 & -30 & -45 & -60 \\ -20 & -40 & -60 & -80 \end{vmatrix}$$

$R_2 - 2R_1;\ R_3 - 3R_1;\ R_4 - 4R_1$

P.T.O

$$|D| = \begin{vmatrix} -5 & -6 & -25 & -20 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 30 & 0 \\ 0 & 0 & 40 & 0 \end{vmatrix}$$

Now

$$|D| = \begin{vmatrix} 12 \\ 30 \\ 40 \end{vmatrix}$$

undefined Determinant.

(v) $\det \begin{bmatrix} 1 & 2 & 3 & n \\ n+1 & n+2 & n+3 & 2n \\ 2n+1 & 2n+2 & 2n+3 & 3n \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & n^2 \end{bmatrix}$

Sol: Let

$$D = \begin{vmatrix} 1 & 2 & 3 & n \\ n+1 & n+2 & n+3 & 2n \\ 2n+1 & 2n+2 & 2n+3 & 3n \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & n^2 \end{vmatrix}$$

$R_2 - 2R_1$; $R_3 - 3R_3$; $R_4 - (R_1)^2$

P.T.O

$$|D| = \begin{vmatrix} 1 & 2 & 3 & n \\ n-1 & n-2 & n-3 & 0 \\ 2n-2 & 2n-4 & 2n-6 & 0 \\ n^2-n & n^2-n-2 & n^2-n-6 & 0 \end{vmatrix}$$

$$|D| = \begin{vmatrix} n-1 & n-2 & n-3 \\ 2n-2 & 2n-4 & 2n-6 \\ n^2-n & n^2-n-2 & n^2-n-6 \end{vmatrix}$$

$$|D| = (n-1)\{(2n-4)\times(n^2-n-6) - (n^2-n-2)\times(2n-6)\}$$
$$- (2n-2)((n-2)\times(n^2-n-6) - (n-3)\times(n^2-n-2))$$
$$+ (n^2-n)((n-2)\times(2n-6) - (2n-4)\times(n-3))$$

$$|D| = (n-1)(2n^3-2n-4n^2+4n+24) - (2n^3-6n^2-2n^2+6n-4n+12)$$
$$-(2n-2)((n^3-n^2-6n-2n^2+2n+12) - (n^3-n^2-2n-3n^2+3n+6))$$
$$+ (n^2-n)((2n^2-6n-\cancel{4n+12}) - (2n^2-6n-\cancel{4n+12}))$$

$$|D| = (n-1)(4n^2+12) - (2n-2)(n^2-5n+6)$$

$$|D| = (4n^3+12n-4n^2-12) - (2n^3-10n^2+12n-2n^2+10n-12)$$

$$|D| = 4n^3+12n-4n^2+2-2n^3+10n^2-12n+2n^2-10n+12$$

$$|D| = 2n^3+8n^2-10n \quad \text{Ans.}$$