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Section :- B

Semester :- 12th Senior

Subject :- Differential Equation

Assignment :- 2

Qno1

The Caculthy equation

1. $x^3 y''' + 2x^2 y' + 2y - 10x + \frac{10}{x}$

Solution

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{dy}{dx} + 2y = 10x + 10x^{-1}$$

$$x^3 D^3 y + 2x^2 D^2 + 2y = 10x + 10x^{-1}$$

$$(x^3 D^3 + 2x^2 D^2 + 2)y = 10x + 10x^{-1} \quad \text{--- (1)}$$

let $x = e^t \Rightarrow t = \ln x$

$$xD = D$$

$$x^2 D^2 = D(\Delta - 1) = \Delta^2 - \Delta$$

$$x^3 D^3 = \Delta(\Delta - 1)(\Delta - 2)$$

Subtituting into Eq (i)

$$(\Delta - 3D^2 + 2D + 2(D^2 - \Delta) + 2)y = 10^2 + 10x^{-1}$$

$$(\Delta^3 - \Delta^2 + 2)y = 10x + 10x^{-1}$$

$$(m^3 - m^2 + 2)y = 10e^t + \frac{10}{e^t}$$

using synthetic division

-1	1	-1	0	2
		-1	2	-2
	1	-2	2	0

Now using Quadratic formula

$a = 1, \; b = -2 \; C = 2$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{-2^2 - 4(1)(2)}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm \sqrt{-1} \times \sqrt{4}}{2}$$

$$= \frac{2 + 2i}{2}$$

$$= \frac{2(1 \pm i)}{2}$$

$$= 1 \pm i$$

Since root are complex

$$y_c = e^{-x}(c_1 \cos t + c_2 \sin t)$$

Now Particular integration

$$Y_P = \frac{1}{D^3-D^2+2} \cdot 10e^t + \frac{1}{D^3-D^2+2} \cdot 10/e^t$$

$$= \frac{10e^t}{(1)^3-(1)^2+2} + \frac{10e^{-t}}{(1)^3-(1)^2+2}$$

$$= \frac{10e^t}{2} + \frac{10e^{-t}}{2}$$

$$= 5e^t + 5e^{-t}$$

$$Y_P = 5e^t + 5e^{-t}$$

General Solution

$$Y = Y_C + Y_P$$

$$Y = e^{-x}(C_1 \cos t + C_2 \sin t) + 5e^t + 5e^{-t}$$

Put $e^t = x$ at $t = \ln x$

$$Y = e^{-x}(C_1 \ln x + C_2 \sin \ln x) + 5e^x + 5e^{-x}$$

Ans.

Q no 2 :-

$$x^3 \frac{d^3y}{dx} + 4x^2 \frac{d^2y}{dx} - 5x\frac{dy}{dx} - 15y = x^4$$

Solution

Let $\frac{d}{dx} = D$

$x^3 D^3 y + 4x^2 D^2 y - 5x Dy - 15y = x^4$

$(x^3 D^3 + 4x^2 D^2 - 5xD - 15)y = x^4$

let $x = e^t \Rightarrow t = \ln x$

$xD = \Delta$

$x^2 D^2 = \Delta(\Delta - 1)\ \Delta^2 - \Delta$

$x^3 D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$

Now Subtituting

$(x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4$

$(\Delta^3 - 3\Delta^2 + 2\Delta + 4(\Delta^2 - \Delta) - 5(\Delta) - 15)$

$y = e^{4t}$

$(\Delta^3 + \Delta^2 - 7\Delta - 15)y = e^{4t}$

Synthetic division

5	1	+1	-7	-15
		3	12	15
	1	4	5	0

$\Delta^2 + 4\Delta 5 = 0$

Formula

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2}$$

$$= \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= \frac{\cancel{2}(-2 \pm i)}{\cancel{2}}$$

$y_c = e^{[illegible]} \, (C_1 \cos t + C_2 \sin t)$

For $y_p = ?$

$$y_p = \frac{1}{D^3 + D^2 - 7D - 15} e^{4t}$$

$$= \frac{1}{(4)^3 + (4)^2 - 7(4) - 15} e^{4t}$$

$$= \frac{1}{64 + 16 - 28 - 15} e^{4t}$$

$$= \frac{1}{80-43} e^{4t}$$

$$y_p = \frac{1}{37} e^{4t}$$

Hence $y = y_c + y_p$

$$y = (C_1 \cos t + C_2 \sin t) + \frac{1}{37} e^{4t}$$

again put $t = \ln x$

and $x = \ln x$

$$y = e^{3x} (C_1 \cos \ln x + C_2 \sin \log x) + \frac{1}{37} e^{4x}$$

Ans

Q no 3

$$x^2 y'' + 2xy' - 6y = 10x^2$$

Solution

$y(1) = 1$ and $y'(1) = -6$

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 6y = 10x^2$$

$$\Rightarrow (x^2 d^2/dx + 2x\, d/dx - 6) y = 10x^2$$

Put $xD = \Delta \Rightarrow x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

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$x = e^t$ and $\log x = t$

$(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}$

The characterstics equation

$\Delta^2 + \Delta - 6 = 0$

$\Delta^2 + 3\Delta - 2\Delta - 6 = 0$

$\Rightarrow \Delta(\Delta+3) - 2(\Delta+3) = 0$

$\Rightarrow (\Delta+3)(\Delta-2) = 0$

$\Delta + 3 = 0$, $\Delta - 2 = 0$

$\Delta = 2$, $\Delta = -3$

Since roots are real and distant for $y_c = ?$

$y_c = C_1 e^{-3t} + C_2 e^{2t}$

for $y_p = ?$

$$y_p = \frac{1}{\Delta^2 - \Delta - 6} 10^{2t}$$

$$= \frac{10}{\Delta^2 - \Delta - 6} e^{2t}$$

$$= 10 \frac{1}{0} e^{2t} \quad \text{fails}$$

Now $10 \dfrac{1}{d/d_0 (\Delta^2 + \Delta - 6)} e^{2t}$

$\Rightarrow 10 \frac{t}{2\Delta+1} e^{2t}$

$= 10 \frac{1-t}{4+1} e^{2t}$

$y_p = 2te^{2t}$

General Equation

$y = y_c + y_p$

$C_1 e^{-3t} + C_2 e^{2t} + 2te^{2t}$

$y = C_1 x^{-3} + C_2 x^2 + 2(\log x) x^2$ —— (B)

Put $y(1) = 1$ i.e $x = 1$, $y = 1$ in (B)

$1 = C_1(1)^{-3} + C_2(1)^2 + 2\log(1)$

$1 = C_1 + C_2 \longrightarrow$ (C)

Now differentiate eq (B)

w.r.t x

$y_1 = 3c_1 x^{-4} + 2(2x + \frac{2}{x}(x)^2 + 4x\log x$

Now put $y'(1) = -6$ i-e $y' = -6$ and $x = -6$

$-6 = -3c_1 + 2C_2 + 2 + 0$

$\Rightarrow -6 = -3C_1 + 2C_2 + 2$

$\Rightarrow -6-2 = 3C_1 + 2C_2 + 2$

$-8 = 3C_1 + 2C_2$ —— (D)

Multiplying Eq 1 with Eq 2 and - from (I

$$2c_1 + 2c_2 = 2$$
$$\mp 3c_1 \pm 2c_2 = \mp 8$$
$$5c_1 = 10$$

$C_1 = \frac{10}{5}$ $\boxed{C_1 = 2}$

$-8 = 3(2) + 2c^2$

$-8 = -6 + 2c^2$

$2c^2 = -8 + 6$

$2c^2 = 2$

$C_2 = \frac{-2}{2}$

$\boxed{C_2 = -1}$

Now Put the values of C_1 and C in Eq (B)

$Y = 2x^{-3} - x^2 + 2 \ln x \; x(x^2)$

$Y = \frac{2}{x^3} - x^2 + 2x^2 \log x$

Ans

Q no 4

$x^2y'' + 7xy' + 5y = x^5$.

$y(0) = 2$ and $y'(1) = 2$

Solution :-

$$x^2\frac{dy^2}{dx^2} + 7x\frac{dy}{dx} + 5y = x^5$$

$$\Rightarrow \left(x^2\frac{d^2}{dx^2} + 7x\frac{d}{dx} + 5\right)y = x^5 \quad \text{—— (A)}$$

Put $xD = \Delta \Rightarrow x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

$x = e^t = \log x = t$ in eq (A)

$$\Rightarrow (\Delta^2 - \Delta - 7\Delta + 5)y = e^{5t}$$

By Quadratic formula

$$\Delta = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36 - 20}}{2}$$

$$= \frac{-6 \pm \sqrt{16}}{2}$$

$$= \frac{6 \pm \sqrt{42}}{2}$$

$$= \frac{\cancel{2}(-3 \pm 2)}{\cancel{2}}$$

$\Delta = -3 \pm 2$ Since roots are real and distant.

$y_c = C_1 e^{-5t} + C_2 e^{-t}$

For $y_P = ?$

$$y_P = \frac{1}{\Delta^2 + 6\Delta + 5} e^{5t}$$

$$= \frac{1}{(5)^2 + 6(5) + 5} e^{5t}$$

$$= \frac{1}{60} e^{5t}$$

Now General Solution is

$y = y_c + y_P$

$y = C_1 e^{-5t} + C_2 e^{-t} + \frac{1}{60} x^5 \rightarrow$ (B)

$x = 0$ put in this equation

Now in eq (B) $e^0 = 1$

Put $y(0) = 2$ i-e $y = 2$ and $x = 2$

$$2 = C_1(2)^{-5} + C_2(2)^{-1} + \frac{1}{60}\cdot(2)^5$$

$$2 = -32C_1 - 2C_2 + \frac{1}{\cancel{60}\,15}(3\overset{8}{\cancel{2}})$$

$$2 = 32C_1 - 2C_2 + \frac{8}{15}$$

$$2 - \frac{8}{15} = -32C_1 - 2C_2$$

$$\frac{22}{15} = -32C_1 - 2C_2 \longrightarrow (C)$$

Now differentiate eq (B) w.r.t (x)

$$y' = -5c_1 x^{-6} - C_2 x^{-2} + \frac{1}{12}x^4$$

Put $y'(1) = 2$ i.e $y' = 2$ and $x = 2$ in above Equation

$$2 = 5c_1\ C_2^{-6} - C_2(2)^{-2} + \frac{1}{12}(2)^4$$

$$2 = -5c_1(-64) - C_2(4) + \frac{1}{12}(16)$$

$$2 = 320C_1 + 4C_2 + 4/3$$

$$\Rightarrow 2 - \frac{4}{3} = 320C_1 + 4C_2$$

$$\Rightarrow 2/3 = 320C_1 + 4C_2 \longrightarrow D$$

Xing Eq(C) with 2 and then − Eq(C) from D

$$\frac{-44}{15} = 64c_1\ 4c^2$$

$$\frac{-44}{15}\ 64c_1 + 4c_1$$

$$\pm 2/3 = \pm 320C_1 \pm 4C_2$$

$$\frac{34}{15} = -256$$

$$C_1 = \frac{34}{15} \times 256$$

$$\boxed{C_1 = 580}$$

Put the value of C_1 in eq (c)

$$\frac{22}{15} = -32(580) - 2C_2$$

$$\frac{22}{15} = -18560 - 2C_2$$

$$\frac{22}{15} + 18560 = 2C_2$$

$$\frac{18561}{-2} = C_2$$

$$\boxed{-9280 = C_2}$$

Now put the values of C_1 and C_2 ing eq (B)

$$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x^5$$

$$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5$$

Ans

Qno5

$$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2$$

$\Rightarrow$ Solution

$$(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2$$

$$\Rightarrow \left((x+1)\frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4\right)y = x^2$$

$$\Rightarrow \left[(x+1)^2D^2 - 3(x+1)D + 4\right]y = x^2 \quad \text{—— (A)}$$

Put $(x+1)D = \Delta \Rightarrow (x+1)^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

$x = e^t$ ing eq (A)

$\Rightarrow [\Delta^2 - \Delta - 3\Delta + 4]y = e^{2t}$

$\Rightarrow [\Delta^2 - 4\Delta + 4)y = e^{2t}$

$\Rightarrow (\Delta^2 - 4D + 4)^2 = e^{2t}$.

for y_c we find the roots

$\Delta^2 - 4\Delta + 4 = 0$

$D^2 - 2\Delta - 2\Delta + 4 = 0$

$\Delta(\Delta - 2) - 2(\Delta - 2) = 0$

$\Delta - 2 = 0 \quad , \Delta = 2$

$\Delta - 2 = 0 \quad ; \Delta = 2$

So the root are real and repeat

The general solution are

$y = (c_1 + c_2 x)^{2x}$

$y = (c_1 + (3x)^{2x}$

for $y_p = ?$

$$y_p = \frac{1}{\Delta^2 - 4\Delta + 4}$$

$$y_p = \frac{2}{2\Delta - 4} e^{2t}$$

if we put 2

$2\Delta - 4 \Rightarrow 2(2) - 4 = 0$

we take again derivative

$y_p = \frac{2}{2} e^{2t}$

$y = (c_1 + (2x)^{2t} + e^{2t}$ Ans