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PAPER * Differential Equation

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QUESTION-No-1
PART 'A'

Differential Equation:-

A differential Equation which contains one or more terms which involve the derivatives of one viarable i.e dependent variable with respect the other variable i.e independent variable.

$$\frac{dy}{dx} = f(x)$$

Here 'x' is independent variable and 'y' is dependent variable.
For Example, $\frac{dy}{dx} = 5x$

Types:-

Differential Equation can be divided into several types

- Ordinary Differential Equations.
- Partial Differential Equations.
- Linear Differential Equations
- Non-Linear Differential Equations.
- Homogenous Differential Equations.
- Non-Homogenous Differential Equation

Solutions:-

To find the solution of differential Equation, There are two methods to solve differential function.

1. Separation of variables.
2. Integrating Factor

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1. **Separation of variable** is done when the differential equation can be written in the form of $\frac{dy}{dx} = F(y)g(x)$ where 'F' is the function of 'y' only and g is the function of x only. Taking an initial condition we rewrite this problem as $1/(F(y))dy = g(x)dx$ and then integrate them from both sides.
2. **Integrating Factor** technique is used when the differential equation is of the form $dy/dx + P(x)y = q(x)$ where 'p' and 'q' are both functions of 'x' only

Examples :-

1. Solve ordianry differential equation $\frac{dy}{dx} = 5x - 3$ for $x(t)$

Sol:- multiply by dt and divide $5x-3$

$$\frac{dx}{5x-3} = dt$$

Integrate both sides

$$\int \frac{dx}{5x-3} = \int dt$$

$$\frac{1}{2} \log |5x - 3| = t + c_1$$

$$\Rightarrow 5x - 3 = \pm \exp(5t + c_1)$$

$$\Rightarrow x = \pm \frac{1}{5} \exp(5t + 5c_1) + 3/5$$

letting $c = \frac{1}{5} \exp(5c_1)$ we can write

$$x(t) = Ce^{5t} + \frac{3}{5}$$

We check to see that $x(t)$ satisfies

$$\frac{dx}{dt} = 5Ce^{5t}$$

$$5x - 3 = 5Ce^{5t} + 3 - 3 = 5Ce^{5t}$$

Both equation are equal verifying our solution.

Example - 2

Solve ODE combined with initial condition

$$\frac{dx}{dt} = 5x - 3 \;, \quad x(2) = 1.$$

Sol:-

$$x(t) = Ce^{5t} + \frac{3}{5}$$

we just need to use the initial condition $x(2) = 1$ to determine c.

$$1 = Ce^{5.2} + \frac{3}{5}$$

$$\Rightarrow C = \frac{2}{5} e^{-10}$$

Our solution is

$$x(t) = \frac{2}{5} e^{5(t-2)} + \frac{3}{5}.$$

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PART 'B'

Separable Differential Equation:

A separabe differential equation is that we can write in following form.

$$N(y) \frac{dy}{dx} = M(x)$$

To solve it, we need to integrate both side with repect 'x'

$$\int N(y) \frac{dy}{dx}\, dx = \int M(x)\, dx.$$

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Now remember y is really $y(x)$ and we can use following substitution.

$$u = y(x), \quad du = y'(x)\,dx = \frac{dy}{dx}\,dx.$$

Applying this substitution to integrate we get

$$\int N(u)\,du = \int M(x)\,dx$$

Finally

$$N(y)\,dy = M(x)\,dx.$$

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"i" Part 'A'

$$y' = \frac{xy^3}{\sqrt{1+x^2}}, \quad y(0) = -1$$

Solution:-

first separate and then integrate both sides

$$y^{-3}\,dy = x(1+x^2)^{-\frac{1}{2}}\,dx$$

$$\int y^{-3}\,dy = \int x(1+x^2)^{-\frac{1}{2}}\,dx$$

$$-\frac{1}{2y^2} = \sqrt{1+x^2} + c$$

Apply the initial condition to get the value of c

$$-\frac{1}{2} = \sqrt{1} + c$$

$$\Rightarrow \quad c = -\frac{3}{2}$$

The implicit solution is then,

$$-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Now let's solve for $y(x)$

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

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$$y^2 = \frac{1}{3-2\sqrt{1+x^2}}$$

$$y(x) = \pm\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

$$y(x) = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}$$

let's get the interval of validity.

$$3-2\sqrt{1+x^2} > 0$$

$$3 > 2\sqrt{1+x^2}$$

$$9 > 4(1+x^2)$$

$$\frac{9}{4} > 1+x^2$$

$$\frac{5}{4} > x^2$$

$$\Rightarrow -\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

The initial condition $x=0$
This interval is therefore our interval of validity.

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"i" Part 'B'

$$y' = e^{-y}(2x-4), \quad y(5)=0$$

Solution:-

first to separate it and integrate both sides

$$e^y \, dy = (2x-4)\, dx$$

Now integrate

$$\int e^y \, dy = \int (2x-4)\, dx$$

$$e^y = x^2 - 4x + c$$

Apply the intitial condition gives

by applying natural log

$$y = \ln(x^2 - 4x + c)$$

we need to find c

$$y(5) = \ln((5)^2 - 4(5) + c)$$

$$\ln|5 + c| = 0$$

$$5 + c = 1$$

$$c = -4$$

$$\Rightarrow y = \ln(x^2 - 4x - 4)$$

Now to find interval of validity

$$x^2 - 4x - 4 > 0$$

The quadratic will be zero at two point

$$x = 2 \pm 2\sqrt{2}$$

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QUESTION-No -2

PART (i) :

STeps of solving linear Differential Equation

Here is a step-by-step method for solving them.

1 Substitute $y = uv$, and

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

into

$$\frac{dy}{dx} + P(x)y = Q(x)$$

2- Factor the parts involving v.

3- Put the v term equal to zero (This gives a differential equation in u and x which can be solved in the next step)

4- Solve using separation of variable to find u.

5- Substitute u back into the equation we got at step 2.

6- Solve that to find v.

7 Finally, substitute u and v into $y = uv$ to get our solution.

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QUESTION-NO-2
PART (ii)

$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$

where $y\left[\frac{\pi}{4}\right] = 3\sqrt{2}$, $0 \leq x \leq \frac{\pi}{2}$

Solution:-

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)}$$

$$y' + \tan(x)y = 2\cos^2(x)\sin(x) - \sec(x)$$

Now find the Integrating factor

$$\mu(t) = e^{\int \tan(x)dx} = e^{\ln|\sec(x)|} = e^{\ln \sec(x)}$$

$$\Rightarrow \sec(x)$$

$$\int \tan(x)dx = -\ln|\cos(x)| = \ln|\cos(x)|^{-1}$$
$$= \ln|\sec(x)|$$

Now,

$$\Rightarrow \sec(x)y' + \sec(x)\tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$$

$$(\sec(x)y)' = 2\cos(x)\sin(x) - \sec^2(x)$$

Integrate both sides

$$\int (\sec(x)y)'\,dx = \int 2\cos(x)\sin(x) - \sec^2(x)$$

$$\Rightarrow \sec(x)y(x) = \int \sin(2x) - \sec^2(x)\,dx$$

$$\Rightarrow \sec(x)\,y(x) = -\frac{1}{2}\cos(2x) - \tan(x) + c$$

Solve for the solution

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + c\cos(x)$$

$$\Rightarrow -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + c\cos(x)$$

Finally

Apply the initial condition to find c

$$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + c\cos\left(\frac{\pi}{4}\right)$$

$$\Rightarrow 3\sqrt{2} = -\frac{\sqrt{2}}{2} + c\,\frac{\sqrt{2}}{2}$$

$$c = 7$$

The solution is

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$$

Answer

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PART (iii):-

$$x' + 2x = \sin t$$

Solution:-

$$x' + P(x)x = Q(x)$$
$$P(x) = 2$$
$$Q(x) = \sin t$$

$$I(x) = e^{\int P(x)dx} = e^{\int 2dt}$$

$$I(x) = e^{2t}$$
$$I(x) = 2t$$

Now,

$$x = \frac{1}{I(x)}\left[\int I(x).Q(x)\,dt + c\right]$$

$$x = \frac{1}{2t}\left[\int 2t.\sin t\,dt + c\right]$$

$$x = \frac{1}{2t}\left[2\int \sin^2 t\,dt + c\right]$$

$$x = \frac{1}{2t}\left[2\int \frac{1}{2}(1-\cos(2t))\,dt + c\right]$$

$$x = \frac{1}{2t}\left[\int 1 - \cos(2t)\,dt + c\right]$$

$$x = \frac{1}{2t}\left[t - \frac{1}{2}\sin(2t) + c\right]$$

$$x = \frac{t}{2t} - \frac{1}{4t}\sin(2t) + \frac{c}{2t}$$

Answer.

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QUESTION-No-3
PART 'A'

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$$
$$y(0) = -3$$

Solution:- First identify M & N

$M = 2xy - 9x^2$
$M_y = 2x$

$= N = 2y + x^2 + 1$
$= N_x = 2x$

So the differential equation is exact

Now,
$\psi_x = M$
$\psi_y = N$

$= \psi = \int M\,dx$ or $\psi \int N\,dy$

$= \psi(x,y) = \int 2xy - 9x^2\,dx$
$= x^2y - 3x^3 + h(y)$

$\psi_y = x^2 + h'(y) = 2y + x^2 + 1 = N$

From this we can see that

$h'(y) = 2y + 1$

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we can now find $h(y)$ by integrating

$$h(y) = \int 2y + 1 \, dy = y^2 + y + K$$

$$\psi(x,y) = x^2y - 3x^2 + y^2 + y + K = y^2 + (x^2+1)y - 3x^3 + K$$

So both K and c are constants

$$= y^2 + (x^2+1)y - 3x^3 = c - K$$

$$= y^2 + (x^2+1)y - 3x^3 - \cancel{K} = c$$

we will not include the K in anymore problem

let's now apply the initial condition to find c

put the value of y & x

$$= (-3)^2 + (0+1)(-3) - 3(0)^3 = c$$

$$\Rightarrow c = 6$$

$$= y^2 + (x^3+1)y - 3x^2 - 6 = 0$$

Now we can solve for $y(x)$ by using quadratic formula.

$$= y(x) = \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4(1)(-3x^3-6)}}{2(1)}$$

$$= y(x) = \frac{(x^2+1) \pm \sqrt{x^4+12x^3+2x^2+25}}{2}$$

reapply the initial condition to figure out which of the two signs in $\pm$ we need

$$= -3 = y(0) = \frac{-1 \pm \sqrt{25}}{2}$$

$$= \frac{-1 \pm 5}{2} = -3, 2$$

we need negative '$-$' sign the implicit solution is then

$$y(x) = \frac{-(x^2+1) - \sqrt{x^4+12x+2x^2+25}}{}$$

Now for interval of validty

$$= x^4 + 12x^3 + 2x^2 + 25 = 0$$

upon solving this equation is zero

at $x = -11.8155\,7624$
and $x = -1.3969\,1133$

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Part 'B' => Question - 3

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0,$$
$$y(5) = 0$$

Solution:- So, first deal with the negative signs separating the two forms.

$$= \frac{2ty}{t^2+1} - 2t + (\ln(t^2+1) - 2)y' = 0$$

Now find M & N

$$= M = \frac{2ty}{t^2+1} - 2t$$

$$= M_y = \frac{2t}{t^2+1}$$

$$= N = \ln(t^2+1) - 2$$

$$= N_y = \frac{2t}{t^2+1}$$

So, it's exact, integrate the first one in this case

$$\psi(t,y) = \int \frac{2t}{t^2+1} - 2t \, dt$$

$$= y\ln(t^2+1) - t^2 + h(y)$$

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Differenite with respect to y and compare to N

$$\psi_y = \ln(t^2+1) + h'(y)$$

$$= \ln(t^2+1) - 2 = N$$

$$h'(y) = -2 \Rightarrow h(y) = -2y$$

This given us

$$= \psi(t, y) = y\ln(t^2+1) - t^2 - 2y$$

$$= y\ln(t^2+1) - t^2 - 2y = c$$

Applying the initial condition gives

$$c = -25$$

$$= y(\ln(t^2+1) - 2) - t^2 = -25$$

$$= y(t) = \frac{t^2 - 25}{\ln(t^2+1) - 2}$$

Interval of validty is

$$= \ln(t^2+1) - 2 = 0$$

$$= \ln(t^2+1) = 2$$

$$= t^2 + 1 = e^2$$

$$= t = \pm\sqrt{e^2 - 1}$$

So

$$\Rightarrow \sqrt{e^2-1} < t < \infty$$

Interval of validity