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PAPER * Differential
Equation

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QUESTION - No - 1 PART 'A'

Differential Equation:-

Equation which contains A differential terms which involve the derivatives of one variable i.e dependent variable with respect the other variable i.e independent variable.

$$\frac{dy}{dx} = f(x)$$

Here 'x' is independent variable and 'y' is dependent variable.

For Example, $\frac{dy}{dx} = 5x$

Types:-

Differential Equation can be divided into several types

- Ordinary Differential Equations.
- Partial Differential Equations.
- Linear Differential Equations
- Non - Linear Differential Equations.
- Homogenous Differential Equations.
- Non - Homogenous Differential Equation

Solutions:-

To find the solution of differential Equation, There are two methods to solve differential function.

- 1 Separation of variables.
- 2 Integrating Factor

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1 Separation of variable is done when the differential equation can be written in the form of $\frac{dy}{dx} = F(y)g(x)$ where 'F' is the function of 'y' only and 'g' is the function of 'x' only.

Taking an initial condition we rewrite this problem as $\int F(y)dy = \int g(x)dx$ and then integrate them from both sides.

2 Integrating Factor technique is used when the differential equation is of the form $dy/dx + P(x)y = q(x)$ where 'P' and 'q' are both functions of 'x' only.

Examples:-

1 Solve ordinary differential equation $\frac{dy}{dx} = 5x - 3$ for $x(t)$

Sol:- multiply by dx and divide $5x-3$

$$\frac{dx}{5x-3} = dt$$

Integrate both sides.

$$\int \frac{dx}{5x-3} = \int dt$$

$$\frac{1}{5} \log |5x-3| = t + c_1$$

$$\Rightarrow 5x-3 = \pm \exp(5t + c_1)$$

$$\Rightarrow x = \pm \frac{1}{5} \exp(5t + 5c_1) + \frac{3}{5}$$

Letting $c = \frac{1}{5} \exp(5c_1)$ we can write

$$x(t) = Ce^{5t} + \frac{3}{5}$$

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We check to see that $x(t)$ satisfies

$$\frac{dx}{dt} = 5Ce^{5t}$$

$$5x - 3 = 5Ce^{5t} + 3 - 3 = 5Ce^{5t}$$

Both equations are equal verifying our solution.

Example - 2

Solve ODE combined with initial condition

$$\frac{dx}{dt} = 5x - 3, \quad x(2) = 1$$

Sol:

$$x(t) = Ce^{5t} + \frac{3}{5}$$

We just need to use the initial condition $x(2) = 1$ to determine C .

$$1 = Ce^{5 \cdot 2} + \frac{3}{5}$$

$$\Rightarrow C = \frac{2}{5} e^{-10}$$

Our solution is

$$x(t) = \frac{2}{5} e^{5(t-2)} + \frac{3}{5}$$

PART 'B'

Separable Differential Equation:

A separable differential equation is that we can write in following form.

$$N(y) \frac{dy}{dx} = M(x)$$

To solve it, we need to integrate both sides with respect to 'x'.

$$\int N(y) \frac{dy}{dx} dx = \int M(x) dx$$

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Now remember y is really $y(x)$ and we can use following substitution.

$$u = y(x), \quad du = y'(x) dx = \frac{dy}{dx} dx.$$

Applying this substitution to integrate we get

$$\int N(u) du = \int M(x) dx$$

Finally

$$N(y) dy = M(x) dx.$$



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"i" Part 'A'

$$y' = \frac{xy^3}{\sqrt{1+x^2}}, \quad y(0) = -1$$

Solution:-

first separate and then
integrate both sides

$$y^{-3} dy = x(1+x^2)^{-\frac{1}{2}} dx$$

$$\int y^{-3} dy = \int x(1+x^2)^{-\frac{1}{2}} dx$$

$$\frac{1}{2y^2} = \sqrt{1+x^2} + c$$

Apply the initial condition
to get the value of c

$$-\frac{1}{2} = \sqrt{1} + c$$

$$\Rightarrow c = -\frac{3}{2}$$

The implicit solution is then,

$$\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Now let's solve for $y(x)$

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

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$$y^2 = \frac{1}{3 - 2\sqrt{1+x^2}}$$

$$y(x) = + \frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

$$y(x) = - \frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

let's get the interval of validity.

$$3 - 2\sqrt{1+x^2} > 0$$

$$3 > 2\sqrt{1+x^2}$$

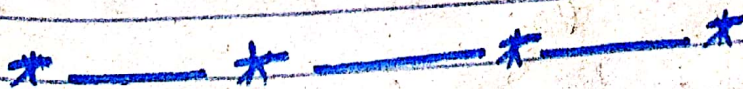
$$9 > 4(1+x^2)$$

$$\frac{9}{4} > 1+x^2$$

$$\frac{5}{4} > x^2$$

$$\Rightarrow -\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

The initial condition $x=0$
This interval is therefore
our interval of validity.



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"i" Part 'B'

$$y' = e^{-y} (2x - 4), \quad y(5) = 0$$

Solution:-

first to separate it
and integrate both sides

$$e^y dy = (2x - 4) dx$$

Now integrate

$$\int e^y dy = \int (2x - 4) dx$$

$$e^y = x^2 - 4x + c$$

Apply the initial condition
gives

by applying natural log

$$y = \ln(x^2 - 4x + c)$$

we need to find c

$$y(5) = \ln(5^2 - 4(5) + c)$$

$$\ln |5 + c| = 0$$

$$5 + c = 1$$

$$c = -4$$

$$\Rightarrow y = \ln(x^2 - 4x - 4)$$

Now to find interval of validity

$$x^2 - 4x - 4 > 0$$

The quadratic will be zero at
two point

$$x = 2 \pm 2\sqrt{2}$$

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QUESTION - NO - 2

PART (i) :

Steps of solving linear Differential Equation

Here is a step-by-step method for solving them.

1. Substitute $y = uv$, and

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

into

$$\frac{dy}{dx} + P(x)y = Q(x)$$

2- Factor the parts involving v .

3- Put the v term equal to zero (This gives a differential equation in u and x which can be solved in the next step)

4- Solve using separation of variable to find u .

5- Substitute u back into the equation we got at step 2.

6- Solve that to find v .

7- Finally, substitute u and v into $y = uv$ to get our solution.



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QUESTION - NO - 2

PART (ii)

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$$

$$\text{where } y\left[\frac{\pi}{4}\right] = 3\sqrt{2}, \quad 0 \leq x \leq \frac{\pi}{4}$$

Solution:

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)}$$

$$y' + \tan(x)y = 2\cos^2(x)\sin(x) - \sec(x)$$

Now find the Integrating factor

$$\mu(x) = e^{\int \tan(x) dx} = e^{\ln|\sec(x)|} = e^{\ln \sec(x)}$$

$$\Rightarrow \sec(x)$$

$$\int \tan(x) dx = -\ln|\cos(x)| = \ln|\cos(x)|^{-1} \\ = \ln|\sec(x)|$$

Now,

$$\Rightarrow \sec(x)y' + \sec(x)\tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$$

$$(\sec(x)y)' = 2\cos(x)\sin(x) - \sec^2(x)$$

Integrate both sides

$$\int (\sec(x)y)' dx = \int 2\cos(x)\sin(x) - \sec^2(x)$$

$$\Rightarrow \sec(x)y(x) = \int \sin(2x) - \sec^2(x) dx$$

$$\Rightarrow \sec(x)y'(x) = -\frac{1}{2}\cos(2x) - \tan(x) + c$$

Solve for the solution

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \cos(x)\tan(x) + c\cos(x)$$

$$\Rightarrow -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + c\cos(x)$$

Finally

Apply the initial condition to find c

$$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + c\cos\left(\frac{\pi}{4}\right)$$

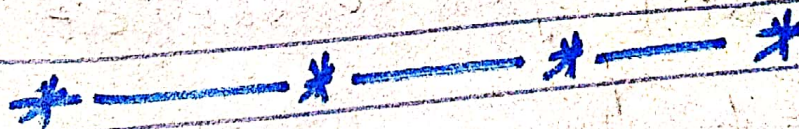
$$\Rightarrow 3\sqrt{2} = -\frac{\sqrt{2}}{2} + c\frac{\sqrt{2}}{2}$$

$$c = 7$$

The solution is

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$$

Answer



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PART (iii):

$$x' + 2x = \sin t$$

Solution:

$$x' + P(x)x = Q(x)$$

$$P(x) = 2$$

$$Q(x) = \sin t$$

$$I(x) = e^{\int P(x) dx} = e^{\int 2 dt}$$

$$I(x) = e^{2t}$$

$$I(x) = 2t$$

Now,

$$x = \frac{1}{I(x)} \left[\int I(x) \cdot Q(x) dt + c \right]$$

$$x = \frac{1}{2t} \left[\int 2t \cdot \sin t dt + c \right]$$

$$x = \frac{1}{2t} \left[2 \int \sin^2 t dt + c \right]$$

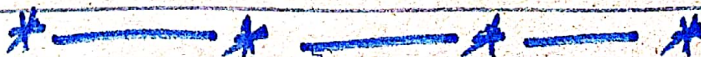
$$x = \frac{1}{2t} \left[2 \int \frac{1}{2} (1 - \cos(2t)) dt + c \right]$$

$$x = \frac{1}{2t} \left[\int 1 - \cos(2t) dt + c \right]$$

$$x = \frac{1}{2t} \left[t - \frac{1}{2} \sin(2t) + c \right]$$

$$x = \frac{t}{2t} - \frac{1}{4t} \sin(2t) + \frac{c}{2t}$$

Answer.



QUESTION - No - 3

PART 'A'

$$2xy - 9x^2 + (2y + x^2 + 1) \frac{dy}{dx} = 0$$

$$y(0) = -3$$

Solution:-

First identify M & N

$$M = 2xy - 9x^2$$

$$M_y = 2x$$

$$= N = 2y + x^2 + 1$$

$$= N_x = 2x$$

So the differential equation is exact

Now,

$$\psi_x = M$$

$$\psi_y = N$$

$$= \psi = \int M dx \quad \text{or} \quad \psi = \int N dy$$

$$= \psi(x, y) = \int 2xy - 9x^2 dx$$

$$= x^2 y - 3x^3 + h(y)$$

$$\psi_y = x^2 + h'(y) = 2y + x^2 + 1 = N$$

From this we can see that

$$h'(y) = 2y + 1$$

We can now find $h(y)$ by integrating

$$h(y) = \int 2y + 1 \, dy = y^2 + y + k$$

$$\psi(x, y) = x^2 y - 3x^2 + y^2 + y + k = y^2 + (x^2 + 1)y - 3x^2 + k$$

So both k and c are constants

$$= y^2 + (x^2 + 1)y - 3x^2 = c - k$$

$$= y^2 + (x^2 + 1)y - 3x^2 = c$$

We will not include the k in anymore problem

Let's now apply the initial condition to find c

Put the value of y & x

$$= (-3)^2 + (0+1)(-3) - 3(0)^2 = c$$

$$\Rightarrow c = 6$$

$$= y^2 + (x^2 + 1)y - 3x^2 - 6 = 0$$

Now we can solve for $y(x)$ by using quadratic formula.

$$= y(x) = \frac{-(x^2 + 1) \pm \sqrt{(x^2 + 1)^2 - 4(1)(-3x^2 - 6)}}{2(1)}$$

$$y(x) = \frac{(x^2 + 1) \pm \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}$$

reapply the initial condition to figure out which of the two signs in \pm we need

$$\begin{aligned} -3 &= y(0) = \frac{-1 \pm \sqrt{25}}{2} \\ &= \frac{-1 \pm 5}{2} = -3, 2 \end{aligned}$$

we need negative '-' sign
the implicit solution is then

$$y(x) = \frac{-(x^2 + 1) - \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}$$

Now for interval of validity

$$x^4 + 12x^3 + 2x^2 + 25 = 0$$

upon solving this equation is zero

$$\text{at } x = -11.81587624$$

$$\text{and } x = -1.396911133$$



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Part 'B' => Question - 3

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0,$$

$$y(5) = 0$$

Solution:- So, first deal with that negative signs separating the two forms.

$$= \frac{2ty}{t^2+1} - 2t + (\ln(t^2+1) - 2)y' = 0$$

Now find M & N

$$= M = \frac{2ty}{t^2+1} - 2t$$

$$= N_y = \frac{2t}{t^2+1}$$

$$= N = \ln(t^2+1) - 2$$

$$= N_y = \frac{2t}{t^2+1}$$

So, it's exact, integrate the first one in this case

$$\Psi(t, y) = \int \frac{2t}{t^2+1} - 2t dt$$

$$= y \ln(t^2+1) - t^2 + h(y)$$

continue on next page

Differentiate with respect to y and compare to N

$$\Psi_y = \ln(t^2+1) + h'(y)$$

$$= \ln(t^2+1) - 2 = N$$

$$h'(y) = -2 \Rightarrow h(y) = -2y$$

This gives us

$$= \Psi(t, y) = y \ln(t^2+1) - t^2 - 2y$$

$$= y \ln(t^2+1) - t^2 - 2y = c$$

Applying the initial condition gives
 $c = -25$

$$= y (\ln(t^2+1) - 2) - t^2 = -25$$

$$= y(t) = \frac{t^2 - 25}{\ln(t^2+1) - 2}$$

Interval of validity is

$$= \ln(t^2+1) - 2 = 0$$

$$= \ln(t^2+1) = 2$$

$$= t^2+1 = e^2$$

$$= t = \pm \sqrt{e^2-1}$$

So

$$\Rightarrow \sqrt{e^2-1} < t < \infty$$

Interval of validity