Name :- Ahmad Ali

ID :- 7746.

Subject :- Differential Equations.

Q1 Find the Fourier Representation of $f(t) = 1+t \quad -\pi \le t \le \pi$.

Sol Given function

$$f(t) = 1+t$$

as we know Fourier Series is given as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin nx$$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

Now finding a_0.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+t)\,dt$$

$$= \frac{1}{\pi}\left[t\Big|_{-\pi}^{\pi} + \frac{t^2}{2}\Big|_{-\pi}^{\pi} \right].$$

$$= \cancel{\frac{1}{\pi}\left[t - t\right] + [}$$

$$= \frac{1}{\pi}\left[\left[\pi-(-\pi)\right] + \frac{1}{2}\left[\pi^2-(-\pi)^2\right]\right]$$

$$a_0 = \frac{1}{\pi}\left[(2\pi) + \left(\frac{1}{2}(\pi^2 - \pi^2)\right)\right]$$

$$= \frac{1}{\pi}\left(2\pi + \frac{1}{2}(0)\right)$$

$$= \frac{2\pi}{\pi}$$

$$\boxed{a_0 = 2}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (1+t)\cos(nt)\,dt)$$

$$= \frac{1}{\pi}\left[1+t\left(\frac{\sin nt}{n}\right)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin(nt)}{n}\right]$$

$$= \frac{1}{\pi}\left[(1+t)\left(\frac{\sin nt}{n}\right) + \frac{\cos nt}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{n(1+t)(\sin nt) + \cos nt}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{ \quad \text{As } \sin n\pi = 0.\right.$$

$$= \frac{1}{\pi}\left[\frac{\cos nt}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi + \cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left(2\frac{\cos n\pi}{n^2}\right)$$

$$= \frac{1}{\pi}\left(2\frac{(-1)^n}{n^2}\right)$$

$$a_n = \frac{2}{\pi n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dx$$

By parts formula.

$$= \frac{1}{\pi}\left[-\frac{(t+1)\cos nt}{n} - \int -\frac{\cos nt}{n}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{(t+1)(\cos nt}{n} - \left(-\frac{\sin nt}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{\sin nt}{n^2} - \frac{t+1(\cos nt)}{n}\right)\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\sin(nt) - n(t+1)(\cos nt)}{n^2}\right)\Bigg|_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left(\frac{-n(t+1)\cos nt}{n^2}\right)\Bigg|_{-\pi}^{\pi}$$ $\because \sin n\pi = 0$

$$= \frac{1}{\pi}\left[\frac{-n(\pi-\pi+1)\cos n\pi - \cos n(-\pi)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-n(\cos n\pi - \cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left(\frac{n(0)}{n^2}\right)$$

$$\boxed{b_n = 0}$$

So Fourier Series of function $1+t$ become

$$\boxed{f(t) = 2 + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}(-1)^n}$$

Scanned with CamScanner

Q2 Given System.

Sol $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix}$

The characteristics equation will be.

$|A - \lambda I| = 0 \quad \text{---} (i)$

$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$.

$A - \lambda I = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$.

$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 3 & 1-\lambda & 4 \\ 0 & 2 & 2-\lambda \end{bmatrix}$

Now $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 3 & 1-\lambda & 4 \\ 0 & 2 & 2-\lambda \end{vmatrix}$

Expanding by R_1.

$= 1-\lambda \begin{vmatrix} 1-\lambda & 4 \\ 2 & 2-\lambda \end{vmatrix} - 0 - 1 \begin{vmatrix} 3 & 1-\lambda \\ 0 & 2 \end{vmatrix}$

$$A-\lambda I = 1-\lambda\left[(1-\lambda)(2-\lambda)-8\right] - 1(6-6)$$

$$= 1-\lambda(2-\lambda-2\lambda+\lambda^2-8)-6$$

$$= 1-\lambda(\lambda^2-3\lambda-6)-6$$

$$= \cancel{\lambda^2-3\lambda-6-\lambda^3+3\lambda^2}$$

as $A-\lambda I = 0$

So

$$1-\lambda(\lambda^2-3\lambda-6)-6=0$$

$$1-\lambda(\lambda^2-3\lambda-6)=6$$

Either

$1-\lambda = 6$ or $\lambda^2-3\lambda-6=6$

when

$$1-\lambda = 6$$

$$\Rightarrow 1-6=\lambda$$

$$\boxed{\lambda = 5}$$

$$\lambda^2-3\lambda-6=6$$

$$\lambda-3\lambda-6-6=0$$

$$\lambda^2-3\lambda-12=0$$

$$\lambda^2-3\lambda=12$$

$$\lambda(\lambda-3)=12$$

$\lambda = 12$

$\lambda - 3 = 12$

$\lambda = 12 + 3$

$\boxed{\lambda = 15}$

So $\lambda = 6$, $\lambda = 12$, $\lambda = 15$ are required Eigen Values.

Q3

Given Equation

$5x + 4z + 2m = 3$

$x - y + 2z + m = 1$

$4x + y + 2z = 1$

$x + y + z + m = 0$.

Sol Writing in Augmented form

~~5 4 2 1~~

$$\left[\begin{array}{cccc:c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 5 & 0 & 4 & 2 & 3 \\ 4 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \quad R_2 - 5R_1$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 0 & 5 & -6 & -3 & -2 \\ 4 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \quad R_3 - 4R_1$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & -6 & -3 & -2 \\ 0 & 5 & -6 & -4 & -3 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

$R_3 + R_2$.

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & -6 & -3 & -2 \\ 0 & 0 & -12 & -7 & -5 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \quad R_4 - R_1$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & -6 & -3 & -2 \\ 0 & 0 & -12 & -7 & -5 \\ 0 & 2 & -1 & 0 & -1 \end{array}\right]$$

$R_4 \to R_4 + \frac{2}{5}R_2$

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ 0 & -5 & -6 & -3 & -2 \\ 0 & 0 & -12 & -7 & -5 \\ 0 & 0 & \frac{-17}{5} & \frac{6}{5} & \frac{-7}{5} \end{array}\right]$$

$R_3 \leftrightarrow R_4$

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ 0 & -5 & -6 & -3 & -2 \\ 0 & 0 & -17/5 & 6/5 & -17/ \\ 0 & 0 & -12 & -7 & -5 \end{array}\right]$$

$R_4 - 2R_2$

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ 0 & -5 & -6 & -3 & 3 \\ 0 & 0 & -17/5 & 6/5 & -17/5 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$\Rightarrow$ $-m = 1$

$\boxed{m = 1}$

$-\frac{17}{5}z + \frac{6}{5}m = -17/5$

$-\frac{17}{5}z = -17/5 - 6/5$

$$\frac{-17}{5}z = \frac{-23}{5}$$

$$z = \frac{-23}{\cancel{5}} \times \frac{\cancel{5}}{-17}$$

$$\boxed{z = \frac{23}{17}}$$

$$-5y - 6z - 3m = 3$$

$$-5y - 6\left(\frac{23}{17}\right) - 3 = 3$$

$$-5y - \frac{138}{17} = 6$$

$$-5y = \frac{17\times 6 + 138}{17}$$

$$= \frac{102 + 138}{17}$$

$$y = \frac{\overset{48}{\cancel{240}}}{17 \times -\cancel{5}}$$

$$\boxed{y = \frac{-48}{17}}$$

$x + y - 2z + m = 1$

$x - \frac{48}{17} - 2\left(\frac{23}{17}\right) + 1 = 1$

$x - \left(\frac{48 - 46}{17}\right) = 0$

$x - \frac{2}{17} = 0$

$$\boxed{x = 2/17}$$

Q4

$u(x,t) = \sin(x + 2t)$

Solution

$$\frac{\partial^2 u}{\partial t^2} = c^2 \partial^2 u / \partial x^2$$

$u(x,t) = \sin(x + 2t)$

$\Rightarrow \frac{\partial u}{\partial t} = \cos \cdot (x + 2t \cdot \frac{\partial}{\partial t} \cdot (x + 2t)$

$= 2 \cos(x + 2t)$

Again differentiating

$$\frac{\partial^2 u}{\partial t^2} = -2 \sin(x + 2t) \frac{\partial}{\partial t}(x + 2t)$$

$$\frac{\partial^2 u}{\partial t^2} = -4\sin(x+2t) \quad -(A)$$

Now $$\frac{\partial u}{\partial x} = \cos(x+2t)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x+2t) \quad -(B)$$

Comparing A & B.

$$-4\sin(x+2t) = -c^2\sin(x+2t$$

$$-4\sin(x+2t) + c^2\sin(x+2t) = 0$$

$$c = \pm 2 \quad \text{is possible.}$$

So $u(x,t) = \sin(x+2t)$

is wave Equation.