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SECTION # "B"

SUBJECT # DIFFERENTIAL EQUATION.

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Q1

(i) $w = \sin(x+ct) + \cos(2x+2ct)$

Sol $w = \sin(x+ct) + \cos(2x+2ct)$

diff w.r.t "t"

$$\frac{dw}{dt} = \cos(x+ct) \times c - \sin(2x+2ct) \times 2c$$

$$\frac{d^2w}{dt^2} = -\sin(x+ct) \times c^2 - \cos(2x+2ct) \times 4c^2$$

$$\frac{d^2w}{dt^2} = c^2(-\sin(x+ct) - 4\cos(2x+2ct)) \quad \text{--- } (1)$$

Now

$w = \sin(x+ct) + \cos(2x+2ct)$

diff w.r.t "x"

$$\frac{dw}{dx} = \cos(x+ct) - \sin(2x+2ct) \times 2$$

Again

$$\frac{d^2w}{dx^2} = -\sin(x+ct) - \cos(2x+2ct) \times 4$$

$$= -\sin(x+ct) - 4\cos(2x+2ct)$$ put in ①

① $\Rightarrow$ $$\frac{d^2w}{dt^2} = c^2 \frac{d^2w}{dt^2}.$$

(ii) $w = \tan x \ (2x + ct)$

Sol

$w = \tan(2x+ct)$

$$\frac{dw}{dx} = \frac{1}{1+(2x+ct)} \times 2 = \frac{2}{1+4x^2+4xct+c^2t^2}$$

$$\frac{d^2w}{d^2x^2} = -2(1+4x^2+4xct+c^2t^2)-2(8x+4c)t$$

$$= \frac{-4(4x+2ct)}{(1+4x^2+4xct+c^2t^2)^2}$$

and

$$\frac{d^2w}{dt^2} = -c/(1+4x^2+4xct+c^2t^2)^2 \times (4x+2c^2t)$$

$$\frac{d^2w}{dt^2} = \frac{-c^2(4x+2ct)}{(1+4x^2+4xct+c^2t^2)^2}.$$

$$\frac{d^2w}{dt^2} = \frac{d_2w}{dx^2} \times c^2/4$$ proved.

Q2 :-

Given function is

$$F(x) = \begin{cases} x; & -\pi < x \le 0 \\ 2x; & 0 \le x \le \pi \end{cases}$$

we have to find the Fourier co-efficients, a_0, a_n & b_n.

Sol Now

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{0} x\,dx + \frac{1}{\pi} \int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi} \left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\cancel{\pi}} \left[0 - \cancel{\pi^2}/2\right] + \frac{\cancel{2}}{\cancel{\pi}} \left[\frac{\cancel{\pi^2}}{2} - 0\right]$$

$$\boxed{a_0 = -\pi/2 + \pi = \pi/2} \rightarrow (1)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} (x \cos nx \, dx) + \frac{1}{\pi} \int_{0}^{\pi} (2x \cos nx) \, dx$$

$$= \frac{1}{\pi} \left[x \left(\frac{\sin nx}{n} \right) - \left(- \frac{\cos nx}{n^2} \right) \right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi} \left[x \left(\frac{\sin nx}{x} \right) - \left(- \frac{\cos nx}{n^2} \right) \right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi} \left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2} \right] + \frac{2}{\pi} \left(\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2} \right)$$

$$= \frac{1}{\pi} \left[\frac{1 - (-1)^n + 2(-1)^n - 2}{n^2} \right] = \frac{(-1)^n - 1}{\pi n^2}.$$

So

$$\boxed{a_n = \begin{cases} -\frac{2}{\pi n^2} & ; \text{ if } n \text{ is add} \\ 0 & ; \text{ if } n \text{ is even} \end{cases}} \quad (2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$\boxed{b_n = \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] = -\frac{3\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}} \rightarrow (3)$$

So the required fourier series is.

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)$$

$$= \pi/4 - 2/\pi\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$$

Q3 Solve the initial vale problem.

$$y'' - 4y + 13y = 8\sin 3x \ , \quad y(0) = 1 \ \& \ y'(0) = 2$$

Sol

$$y'' - 4y + 13y = 8\sin 3x \quad \text{— (1)}$$

Homogenous eq. of (1) is

$$y'' - 4y' + 13y = 0 \quad \text{— (2)}$$

Change equ (ii) into Auxiliary equation

put $y = m$ — in (2)

(2) $\Rightarrow$ $m^2 - 4m + 13 = 0$

Use Quadratic formula.

$a = 1, \quad b = -4 \ , \ c = 13.$

8

$$m \quad \frac{-b \pm \sqrt{b^2-4ac}}{2}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2-4(1)(13)}}{2(1)}$$

$$m = \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm \sqrt{36i}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$m = \pm 3i$$

$$\boxed{\begin{aligned} m_1 &= 2+3i \\ m_2 &= 2-3i \end{aligned}}$$

$y_c = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$ — (ii)

let

$y_P = A \cos 3x + B \sin 3x$ — (x)

Diff w.r.t "x"

$y'_P = -3A \sin 3x + 3B \cos 3x$

Again diff w.r.t "x"

$y''_P = -9A \cos 3x - 9B \sin 3x \rightarrow$ put in ①

① $\Rightarrow (-9A \cos 3x - 9B \sin 3x) - 4(-3A \sin 3x + 3B \cos 3x)$
$+ 13 (A \cos 3x + B \sin 3x) = 8 \sin 3x$

$\Rightarrow \quad -9A \cos 3x - 12B \cos 3x + 13A \cos 3x - 9B \sin x +$
$12A \sin 3x + 13B \sin 3x \} = 8 \sin 3x.$

$\Rightarrow \quad (4A - 12B) \cos 3x + (4B + 12A) \sin 3x = 8 \sin 3x.$

Comparing cofficients.

$\sin 3x \Rightarrow \quad 4B + 12A = 8 \rightarrow (a)$

$\cos 3x \Rightarrow \quad 4A - 12B = 0 \Rightarrow 4A = 12B$

$A = 3B \rightarrow (B).$

put (B) in (a)

$4B + 12(3B) = 8$

$4B + 36(B) = 8$

$40B = 8$

$B = 1/5 \quad - (C)$

put (C) in b

$A = 3/5 \rightarrow (d)$

put "c" and "d" in x.

$y_P = B/5 \cos 3x + 1/5 \sin 3x \quad - (B).$

The general solution is

$y = y_c + y_P.$

(11)

$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + 3/5 \cos 3x + 1/5 \sin 3x$ — (C)

Need to find the vale of

C_1 & C_2 for this put $x = 0$

& $y = 1$ in (C)

(C) $\Rightarrow$ $1 = e^{2(0)}\left(c_1 \cos(30) + c_2 \sin 3(0)\right) + 3/5 \cos 3(0)$
$+ 1/5 \sin 3(0)$

$1 = \left(1(1) + (2(0)) + 3/5(1) + 1/5(0)\right)$

$1 = C_1 + 3/5$

$C_1 = 1 - 3/5$

$C_1 = 2/5 \rightarrow (C_0)$

Diff C w.r.t "x"

(12)

$$y' = C_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + C_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x - 6/5 \sin 3x + 3/5 \cos 3x$$ (D)

put $y' = 2$, $x = 0$ in "D"

$y' = 2$, $x = 0$ in "D"

$$y' = C_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x)$$
$$+ C_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x) - 6/5 \sin 3x + 3/5 \cos 3x$$

put $y' = 2$, $x = 0$.

$$2 = C_1(2e^{2(0)}\cos 3(0) - 3e^{2(0)}\sin 3(0)) +$$
$$C_2(2e^{2(0)}\sin 3(0) + 3e^{2(0)}\cos 3(0)) - 6/5 \sin$$
$$3(0) + 3/5 \cos 3(0)$$

$$2 = C_1(2) + C_2(3) - 0, + 3/5$$

$$2 = 2C_1 + 3C_2 + 3/5$$

put $C_1 = 2/5$

$$2 = 4/5 + 3C_2 + 3/5$$

$$2 = 4/5 + 3C_2 + 3/5$$

(13)

$$2 = 7/5 + 3c_2$$

$$3c_2 = 2 - 7/5$$

$$3c_2 = 3/5$$

$$\boxed{c_2 = 3/15} \quad x_3$$

put x_2 & x_3 in "c"

$$y = e^{2x}(2/5 \cos 3x + 3/15 \sin 3x) + 3/5 \cos 3x + 1/5 \sin 3x.$$

$$\boxed{y = 2/5\, e^{2x} \cos 3x + 3/15\, e^{2x} \sin 3x + 3/5 \cos 3x + 1/5 \sin 3x}$$

Required general formula.

Q4:- $(D^2 - DD')z = \cos x \cos 2y$

Sol

$(D^2 - DD')z = \cos x \cos 2y$

The given PDE can be rewrite as

$D(D-D')u = \cos x \cos 2y$

In CF is given by:

$CF = \phi_1(y) + \phi_2(y+x)$

While its PI is given by :-

$PI = \frac{1}{(D^2 - DD')} \cdot \frac{1}{2}\left[\cos(x+2y) + \cos(x-2y)\right]$

$= \frac{1}{2}\left[\frac{1}{(-1+2)}\cos(x+2y) + \frac{1}{(-1-2)}\cos(x-2y)\right]$

$= \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$

Hence the complete solution of the given PDE is given by.

$$u = \phi_1(y) + \phi_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y).$$