Deptt: Civil Engineering

Exam:	Mid term
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①

Q-1: Solve The following objective Type questions.

(i) The order of matrix A is $m \times p$ and The order of B is $p \times m$. Then order of matrix AB is $\underline{m \times m}$.

$\left[\begin{array}{ll} A & B \\ m\times p & p\times m \end{array}\right.$ order of AB $= m\times m$

(ii) The number of non-zero rows in an Echelon form is called $\underline{\text{Rank of Matrix}}$.

(iii) If $B = \begin{bmatrix} 1 & 4 \\ 2 & a \end{bmatrix}$ is a singular Matrix Then $a = \underline{8}$

Sol: Now B is singular Matrix. Therefore

$\det B = 0$

$\Rightarrow \begin{vmatrix} 1 & 4 \\ 2 & a \end{vmatrix} = 0$

$\Rightarrow 1\cdot a - 2\cdot 4 = 0$

$\Rightarrow a - 8 = 0$

$\Rightarrow a = 8$

(iv) If $A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$ is $\underline{\text{Scalar}}$ Matrix.

Definition A square matrix A is scalar if $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ where k is a scalar and $k \neq 0$.

(iv) if $A = \begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$ Then $|A| = \underline{3}$

Sol: $A = \begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$

$|A| = \begin{vmatrix} 2i & i \\ i & -i \end{vmatrix}$

$|A| = 2i(-i) - i\cdot i$

$= -2i^2 - i^2$ [Put $i^2 = -1$]

$= -2(-1) - (-1)$

$= 2 + 1$

$|A| = 3$

(vi) Solution of $\frac{dy}{dx} + 2xy = y$?

Sol:: $\frac{dy}{dx} + 2xy = y$

$\Rightarrow \frac{dy}{dx} = y - 2xy$

$\Rightarrow \frac{dy}{dx} = y(1-2x)$

By variable Seprable

$\Rightarrow \frac{dy}{y} = (1-2x)dx$

Integrate both sides

$\Rightarrow \int \frac{1}{y} dy = \int (1-2x)dx$

$\Rightarrow \ln y = \int 1 dx - 2\int x dx$

$\Rightarrow \ln y = x - 2\frac{x^{1+1}}{1+1} + c$

$\Rightarrow \ln y = x - 2\frac{x^2}{2} + c$

$\Rightarrow \log_e y = (x - x^2) + c$

Taking Anti log

$\Rightarrow y = e^{(x-x^2)+c}$

$\Rightarrow y = e^{x-x^2} \cdot e^{c}$

Put $e^c =$ Constant $= c_1$

$\Rightarrow y = e^{x-x^2} \cdot c_1$.

Hence Solution of $\frac{dy}{dx} + 2xy = y$ is, $\underline{y = e^{x-x^2} \cdot c_1}$

(vii) The order and degree of differential equation.

$$\left(\frac{dy}{dx}\right)^3 = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$ is. $\left[\begin{array}{l}\text{order} = 01 \\ \text{Degree} = 6\end{array}\right]$

Sol:: **ORDER** :: The order of a differential equation is the order of the highest derivative appearing in it.

Degree:: The exponent of highest-order derivative is called the degree of the differential equation.

Given Problem:: $\left(\frac{dy}{dx}\right)^3 = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Taking squaring both sides.

$$\Rightarrow \left[\left(\frac{dy}{dx}\right)^3\right]^2 = \left[\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right]^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^6 = 1+\left(\frac{dy}{dx}\right)^2$$

$\Rightarrow$ Order = I (one)

and Degree = 6

(viii) The order and degree of differential equation.

$$\frac{d^2y}{dx^2} - 4xy = \sin\left(\frac{d^2y}{dx^2}\right)$$ is ?

Order = 2

degree = not define.

[As maculerine Expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin\left[\frac{d^2y}{dx^2}\right] = \left(\frac{d^2y}{dx^2}\right)^1 - \frac{\left(\frac{d^2y}{dx^2}\right)^3}{3!} + \frac{\left(\frac{d^2y}{dx^2}\right)^5}{5!} - \frac{\left(\frac{d^2y}{dx^2}\right)^7}{7!} + \dots$$

degree = 1 or 3, 5, 7, ... (not define).]

(ix) The differential equation $2\frac{dy}{dx} + x^2y = 2x+3$, $y(0) = 5$ is ?

is Linear differential equation of first order.

Sol: $2\frac{dy}{dx} + x^2y = 2x+3 \rightarrow (1)$, $y(0) = 5$

Deviding by 2

$\Rightarrow \frac{dy}{dx} + \frac{1}{2}x^2y = \frac{2x}{2} + \frac{3}{2}$

$\Rightarrow \frac{dy}{dx} + \frac{1}{2}x^2y = x + \frac{3}{2} \longrightarrow (2)$

$\Rightarrow \frac{dy}{dx} + P(x)y = Q(x)$ [General form of Linear diff: Equation

Integrating factor $= e^{\int P(x)dx}$ $=$ (Put $P(x) = \frac{1}{2}x^2$)

$I.F = e^{\frac{1}{2}\int x^2dx} = e^{\frac{1}{2}\frac{x^{2+1}}{2+1}} = e^{\frac{1}{2}\frac{x^3}{3}} = e^{\frac{x^3}{6}}$

Multiply (2) by $e^{\frac{x^3}{6}}$

$\Rightarrow e^{\frac{x^3}{6}}\frac{dy}{dx} + \frac{1}{2}e^{\frac{x^3}{6}}.x^2y = e^{\frac{x^3}{6}}(x+\frac{3}{2})$

$\Rightarrow \frac{d}{dx}[e^{\frac{x^3}{6}}.y] = e^{\frac{x^3}{6}}(x+\frac{3}{2})$

Integrate both sides.

$\Rightarrow d[e^{\frac{x^3}{6}}.y] = e^{\frac{x^3}{6}}(x+\frac{3}{2})dx$

Integrate both sides.

$\Rightarrow \int d[e^{\frac{x^3}{6}}.y] = \int e^{\frac{x^3}{6}}(x+\frac{3}{2})dx$

$\Rightarrow e^{\frac{x^3}{6}}y = \int e^{\frac{x^3}{6}}(x+\frac{3}{2})dx$

[Integration is not Possible]

P.T.O

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Maculisine ExPansion of

$$e^u = 1+u+\frac{u^2}{2!}+\cdots$$

Put $u = \frac{x^3}{6}$

$$e^{\frac{x^3}{6}} = 1+\frac{x^3}{6}+\frac{\left(\frac{x^3}{6}\right)^2}{2\cdot 1}+\cdots$$

$$= 1+\frac{x^3}{6}+\frac{x^6}{72}+\cdots$$

$$e^{\frac{x^3}{6}}\cdot y = \int e^{\frac{x^3}{6}}\left(x+\frac{3}{2}\right)dx$$

Put $e^{\frac{x^3}{6}} = 1+\frac{x^3}{6}+\frac{x^6}{72}+\cdots$

$$e^{\frac{x^3}{6}}\cdot y = \int \left(x+\frac{3}{2}\right)\left(1+\frac{x^3}{6}+\frac{x^6}{72}+\cdots\right)dx$$

$$e^{\frac{x^3}{6}}\cdot y = \int\left[\left(x+\frac{x^4}{6}+\frac{x^7}{72}+\cdots\right)+\left(\frac{3}{2}+\frac{3}{2}\cdot\frac{x^3}{6_2}+\frac{3}{2}\cdot\frac{x^6}{72_{24}}+\cdots\right)\right]dx$$

$$e^{\frac{x^3}{6}}\cdot y = \int\left(\frac{3}{2}+x+\frac{x^3}{4}+\frac{x^4}{6}+\frac{x^6}{48}+\frac{x^7}{72}+\cdots\right)dx$$

$$e^{\frac{x^3}{6}}\cdot y = \left(\frac{3}{2}x+\frac{x^2}{2}+\frac{x^4}{4\cdot 4}+\frac{x^5}{5\cdot 6}+\cdots\right) + C$$

$$e^{\frac{x^3}{6}}\cdot y = \left(\frac{3}{2}x+\frac{x^2}{2}+\frac{x^4}{16}+\frac{x^5}{36}+\cdots\right) + C \rightarrow (3)$$

Put $x=0,\ y=5$

$$e^0\cdot 5 = 0+0+C$$

$$1\cdot 5 = C$$

$$\Rightarrow C = 5$$

Put $C=5$ in (3)

$$e^{\frac{x^3}{6}}\cdot y = \left(\frac{3}{2}x+\frac{x^2}{2}+\frac{x^4}{16}+\frac{x^5}{36}+\cdots\right)+5.$$

(x): $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ (B) is $\underline{(a-b)(b-c)(c-a)}$.

Sol:
$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} \quad \begin{matrix} R_1 - R_2 \\ R_2 - R_3 \end{matrix}$$

Expand by C_1 (First column)

$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

$= 0 - 0 + 1 \cdot \begin{vmatrix} a-b & a^2-b^2 \\ b-c & b^2-c^2 \end{vmatrix}$

$= \begin{vmatrix} a-b & (a-b)(a+b) \\ b-c & (b-c)(b+c) \end{vmatrix}$

Taking Common $(a-b)$ from R_1 and $(b-c)$ from R_2

$= (a-b)(b-c)\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix}$

$= (a-b)(b-c)\left[1(b+c) - 1\cdot(a+b)\right]$

$= (a-b)(b-c)\left[\cancel{b}+c-a-\cancel{b}\right]$

$= (a-b)(b-c)(c-a)$

Q. No: 2 ::

(i) Express The determinant $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ as the Product of factor wich are linear in a, b, c.

Sol:

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix} \quad \text{by } \begin{matrix} C_1 - C_2 \\ C_2 - C_3 \end{matrix}$$

$$= \begin{vmatrix} (a-b) & (b-c) & c \\ (a-b)(a+b) & (b-c)(b+c) & c^2 \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

Taking Common $(a-b)$ from C_1 and $(b-c)$ from C_2.

$$= (a-b)(b-c) \begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & c \\ a+b-b-c & b+c & c^2 \\ a^2+ab+\cancel{b^2}-\cancel{b^2}-bc-c^2 & b^2+bc+c^2 & c^3 \end{vmatrix} \quad C_1 - C_2$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & c \\ a-c & b+c & c^2 \\ a^2-c^2+ab-bc & b^2+bc+c^2 & c^3 \end{vmatrix}$$

(8)

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & c \\ a-c & b+c & c^2 \\ (a-c)(a+c)+b(a-c) & b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & c \\ a-c & b+c & c^2 \\ (a-c)(a+c+b) & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Taking Common $(a-c)$ from C_1.

$$= (a-b)(b-c)(a-c)\begin{vmatrix} 0 & 1 & c \\ 1 & b+c & c^2 \\ a+b+c & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Expand by R_1

$$= (a-b)(b-c)(a-c)\left[a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}\right]$$

$$= (a-b)(b-c)(a-c)\left[0 - 1\begin{vmatrix} 1 & c^2 \\ a+b+c & c^3 \end{vmatrix} + c\begin{vmatrix} 1 & b+c \\ a+b+c & b^2+bc+c^2 \end{vmatrix}\right]$$

$$= (a-b)(b-c)(a-c)\left[-\{c^3 - c^2(a+b+c)\} + c\{1(b^2+bc+c^2) - (b+c)(a+b+c)\}\right]$$

$$= (a-b)(b-c)(a-c)\left[-c^3 + c^2(a+b+c) + c(b^2+bc+c^2) - c(b+c)(a+b+c)\right]$$

$$= (a-b)(b-c)(-c+a)\left[-c^3 + c^2(a+b+c) + c(b^2+bc+c^2) - (bc+c^2)(a+b+c)\right]$$

$$= (a-b)(b-c)(-1)(c-a)\left[-c^3 + c^2\cancel{(a+b+c)} + c(b^2+bc+c^2) - bc(a+b+c) - c^2\cancel{(a+b+c)}\right]$$

$$= (a-b)(b-c)(c-a)(-1)\left[-\cancel{c^3} + \cancel{cb^2} + \cancel{bc^2} + \cancel{c^3} - abc - \cancel{b^2c} - \cancel{bc^2}\right]$$

$$= (a-b)(b-c)(c-a)(-1)(-abc)$$

$$= (a-b)(b-c)(c-a)\,abc$$

$$= abc\,(a-b)(b-c)(c-a)$$

Q.NO:2 (9)

(ii) Find The Eigam value of $\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$

Sol:. Let $A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Let λ is Eigam value of A.

then Characteristic Equation is

$$\det[A - \lambda I] = 0$$

$$\det\left(\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 2-\lambda & -1 & -1 & 0 \\ -1 & 3-\lambda & -1 & -1 \\ -1 & -1 & 3-\lambda & -1 \\ 0 & -1 & -1 & 2-\lambda \end{bmatrix} = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & -1 & 0 \\ -1 & 3-\lambda & -1 & -1 \\ -1 & -1 & 3-\lambda & -1 \\ 0 & -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 0 & 0 \\ -1 & 3-\lambda & -4+\lambda & -1 \\ -1 & -1 & 4-\lambda & -1 \\ 0 & -1 & 0 & 2-\lambda \end{vmatrix} \; C_3 - C_2 \quad = 0$$

(10)

$$\begin{vmatrix} 2-\lambda & -1 & 0 & 0 \\ -1 & 3-\lambda & -(4-\lambda) & -1 \\ -1 & -1 & 4-\lambda & -1 \\ 0 & -1 & 0 & 2-\lambda \end{vmatrix} = 0$$

Taking Common $(4-\lambda)$ from C_3

$$(4-\lambda)\begin{vmatrix} 2-\lambda & -1 & 0 & 0 \\ -1 & 3-\lambda & -1 & -1 \\ -1 & -1 & \cancel{-}1 & -1 \\ 0 & -1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\begin{vmatrix} 2-\lambda & -1 & 0 & 0 \\ -2 & 2-\lambda & 0 & -2 \\ -1 & -1 & 1 & -1 \\ 0 & -1 & 0 & 2-\lambda \end{vmatrix} \; R_2+R_3 \quad = 0$$

Expand by C_3

$(4-\lambda)\left[a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} + a_{43}A_{43} \right] = 0$

$$(4-\lambda)\left[0 - 0 + 1 \begin{vmatrix} 2-\lambda & -1 & 0 \\ -2 & 2-\lambda & -2 \\ 0 & -1 & 2-\lambda \end{vmatrix} - 0 \right] = 0$$

Expand by R_1

$$(4-\lambda)\left[(2-\lambda)\begin{vmatrix} 2-\lambda & -2 \\ -1 & 2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -2 & -2 \\ 0 & 2-\lambda \end{vmatrix} + 0 \right] = 0$$

$(4-\lambda)\left[(2-\lambda)\{(2-\lambda)^2 - 2\} + \{-2(2-\lambda) - 0\} \right] = 0$

$(4-\lambda)(2-\lambda)\left[(2-\lambda)^2 - 2 - 2\right] = 0$

$(4-\lambda)(2-\lambda)\left[\cancel{4} + \lambda^2 - 4\lambda - \cancel{4}\right] = 0$

$(4-\lambda)(2-\lambda)(\lambda)(\lambda-4) = 0$

$\Rightarrow 4-\lambda = 0$, $4 = \lambda$, $\lambda = 4$ | $2-\lambda = 0$, $2 = \lambda$, $\lambda = 2$ | $\lambda = 0$ | $\lambda - 4 = 0$, $\lambda = 4$

Eigam value are $\lambda = 0, 2, 4$

Q. No: 3 :: The of Change in the form of differential equation is given by, $(x^2+3y^2)dx - 2xy\,dy = 0$ Find the general solution at $x=2$ and $y=6$

Sol::
$$(x^2+3y^2)dx - 2xy\,dy = 0$$
$$\Rightarrow (x^2+3y^2)dx = 2xy\,dy$$
$$\Rightarrow 2xy\,dy = (x^2+3y^2)dx$$
Dividing both sides by $2xy\,dx$
$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+3y^2)}{2xy} \longrightarrow \text{(I)}$$
It is a homogenouse Differential equation.

Put $y = vx \longrightarrow (2)$

Diff: w-r-t. x
$$\frac{dy}{dx} = \frac{d}{dx}(v.x) \quad \text{(Using Product Rule)}$$
$$\frac{dy}{dx} = v.\frac{d}{dx}x + x\frac{dv}{dx}$$
$$\frac{dy}{dx} = v.1 + x\frac{dv}{dx}$$
$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$
$\Rightarrow$ Now Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (I). we get
$$v + x\frac{dv}{dx} = \frac{x^2+3(vx)^2}{2.x.vx}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^2+3v^2x^2}{2x^2v}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{\cancel{x^2}(1+3v^2)}{2\cancel{x^2}.v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1+3v^2-2v^2}{2v}$$

$\Rightarrow x \dfrac{dv}{dx} = \dfrac{1+v^2}{2v}$

Now by Veriable SePrable

$\Rightarrow \dfrac{2v}{1+v^2} dv = \dfrac{dx}{x}$

Integrate both Sides

$\Rightarrow \int \dfrac{2v}{1+v^2} dv = \int \dfrac{1}{x} dx$

[By using $\int \dfrac{f'(x)}{f(x)} dx = \ln f(x)$]

$\Rightarrow \ln(1+v^2) = \ln x + \ln c$ (where $\ln c$ is Constant of Integration).

$\Rightarrow \ln(1+v^2) = \ln x \cdot c$ [As $\log m + \log n = \log mn$].

Taking Ant ln both Sides

$\Rightarrow (1+v^2) = xc \rightarrow (3)$

From (2) Put $v = \dfrac{y}{x}$ in (3)

$\Rightarrow 1 + \left(\dfrac{y}{x}\right)^2 = xc$

$\Rightarrow 1 + \dfrac{y^2}{x^2} = xc$

Multiply by x^2

$\Rightarrow x^2 + y^2 = x^3 c \rightarrow (4)$

Apply The initial Conditions. Put $x = 2$, $y = 6$ in (4)

$\Rightarrow (2)^2 + (6)^2 = (2)^3 \cdot c$

$\Rightarrow 4 + 36 = 8c$

$\Rightarrow 40 = 8c$

$\Rightarrow c = \dfrac{40}{8} = 5$

Put $c = 5$ in (4)

$x^2 + y^2 = x^3 \cdot 5$

$\Rightarrow y^2 = 5x^3 - x^2$

$\Rightarrow \sqrt{y^2} = \sqrt{5x^3 - x^2}$

$y = \pm\sqrt{5x^3 - x^2} \Rightarrow \boxed{y = +\sqrt{5x^3 - x^2}}$