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Course Numerical Analysis:

Q1 (a) Find the LU Factorization of the matrix.

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{pmatrix}.$$

Sol: The elimination steps proceed as.

$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{pmatrix}$ subtract $2\times$ row 1 from row 2 $\begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ -3 & 1 & 1 \end{pmatrix}$

subtract $-3 \times$ row 1 from row 3 $\begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 7 & -2 \end{pmatrix}$

subtract $-7/3 \times$ row 2 from row 3 $\begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix} = U$

By putting 1's on the main diagonal and the multipliers in the lower triangle in the specific places they were used for elimination. that B.

Q1(a)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -7/3 & 1 \end{bmatrix}.$$

Now check that.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -7/3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{pmatrix} = A.$$

The reason that this procedure gives the LU factorization follows from three facts about lower triangular matrices.

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Q 1(b) Find the unknown variables of the following equation using gaussian elimination methods.

$$x + 2y - z = 3$$
$$2x + y - 2z = 3$$
$$-3x + y + z = -6$$

Sol:- The tableau form as.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 1 & -2 & 3 \\ -3 & 1 & 1 & -6 \end{array}\right]$$

Two steps are needed to eliminate ~~column~~ column 1:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 1 & -2 & 3 \\ -3 & 1 & 1 & -6 \end{array}\right)$$ subtract $2 \times$ row 1 from row 2 $$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ -3 & 1 & 1 & -6 \end{array}\right]$$

Subtract $-3 \times$ row 1 from row 3 $$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 7 & -2 & 3 \end{array}\right)$$

and one more step to eliminate column 2.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 7 & -2 & 3 \end{array}\right]$$ subtract $-7/3 \times$ row 2. from row 3 $\rightarrow$ $$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & -2 & -4 \end{array}\right)$$

Returning to the equations:

Q1 (b)

$$x + 2y - z = 3.$$
$$-3y = -3.$$
$$-2z = -4$$

We can solve for the variables.

$$x = 3 - 2y + z.$$
$$-3y = -3.$$
$$-2z = -4.$$

The solution is $x = 3$, $y = 1$, $z = 2$.

Q1 (c) Apply Gaussian elimination with partial pivoting to solve the following system of equations.

$$x_1 - x_2 + 3x_3 = -3.$$

$$-x_1 - 2x_3 = 1.$$

$$2x_1 + 2x_2 + 4x_3 = 0.$$

Soln

This example is written in tableau form as.

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 2 & 4 & 0 \end{array}\right].$$

under partial pivoting we compare $|a_{11}| = 1$. with $|a_{21}| = 1$ and $|a_{31}| = 2$ and chose a_{31} for the new pivot. This is achived through an exchange of rows 1 and 3.

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 2 & 4 & 0 \end{array}\right] \longrightarrow \text{exchage row 1 and row 3} \rightarrow \left(\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right).$$

Sub. $-\frac{1}{2} \times$ row 1. from row 2. $\rightarrow$
$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

sub $\frac{1}{2} \times$ row 1 from row 3 $\rightarrow$
$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array}\right].$$

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Before eliminating column 2 we must compare the current $|a_{22}|$ with the current $|a_{32}|$. Because the latter is larger, we again switch rows.

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array}\right] \xrightarrow{\text{exchange row 2 and row 3}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$$

Sub $-\frac{1}{2} \times$ row 1

$\longrightarrow$ from row 2 $\rightarrow$ $\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 3 & -3 \end{array}\right]$.

Sub $\frac{1}{2} \times$ row 1.

from row 3 $\rightarrow$ $\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array}\right]$.

Before eliminating column 2 we must compare the current $|a_{22}|$ with the current $|a_{32}|$. Because the latter is larger, we again switch rows.

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -3 \end{array}\right] \xrightarrow{\text{exchange row 2 and row 3}} \left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & -2 & 1 & -3 \\ 0 & 1 & 0 & 1 \end{array}\right].$$

Sub $-\frac{1}{2} \times$ row 2

from row 3 $\rightarrow$ $\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$

Q1(c)

Note that all three multipliers are less than 1 in absolute Value.
The equation are now simple to Solve. from

$$\frac{1}{2} x_3 = -\frac{1}{2}.$$

$$-2x_2 + x_3 = -3$$

$$2x_1 + 2x_2 + 4x_3 = 0.$$

we find that $x = [1, 1, -1]$.

Q 2 (A) Apply Gauss-seidal Method to the following system.

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 4 & 1 \\ -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

Sol: The Gauss-seidal itteration is

$$u_{k+1} = \frac{4 - v_k + w_k}{3}.$$

$$v_{k+1} = \frac{1 - 2u_{k+1} - w_k}{4}.$$

$$w_{k+1} = \frac{1 + u_{k+1} - 2v_{k+1}}{5}.$$

Starting with $x_0 = [u_0, v_0, w_0] = [0, 0, 0]$.

we calculate.

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \frac{4-0-0}{3} = 4/3. \\ 1 - 8/3 - 0/4 = -5/12. \\ \frac{1 + 4/3 + 5/6}{5} = 19/20 \end{bmatrix} = \begin{bmatrix} 1.333 \\ -0.4167 \\ 0.6333 \end{bmatrix}.$$

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{101}{60} \\ -3/4 \\ 251/300 \end{bmatrix} \approx \begin{bmatrix} 1.6833 \\ -0.7500 \\ 0.8367 \end{bmatrix}.$$

The system is strictly diagonally dominant, and therefore the iteration will converge to the solution $[2, -1, 1]$.

Q2 (b). Find the reduced QR factorization by applying Gram-Schmidt orthogonalization to the column of the following matrix.

$$A = \begin{bmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{bmatrix}.$$

Sol:

set $y_1 = A_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ Then $r_{11} = \|y_1\|_2 = \sqrt{1^2+2^2+2^2} = 3.$

and the first unit vector is

$$q_1 = \frac{y_1}{\|y_1\|_2} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}.$$

To find the second unit vector, set

$$y_2 = A_2 - q_1 q_1^T A_2 = \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} 2 = \begin{bmatrix} -14/3 \\ 5/3 \\ 2/3 \end{bmatrix}.$$

and $$q_2 = \frac{y_2}{\|y_2\|_2} = \frac{1}{5}\begin{bmatrix} -14/3 \\ 5/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -14/15 \\ 1/3 \\ 2/15 \end{bmatrix}.$$

Since $r_{12} = q_1^T A_2 = 2$ and $r_{22} = \|y_2\|_2 = 5$

$$A = \begin{bmatrix} 1 & -4 \\ 2 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1/3 & -14/15 \\ 2/3 & 1/3 \\ 2/3 & 2/15 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 5 \end{bmatrix} = QR.$$

Q2(c) let $x = \begin{bmatrix} 3 & 4 \end{bmatrix}$ and $w = \begin{bmatrix} 5 & 0 \end{bmatrix}$. 14
find a house holder reflector that satisfies $Hx = w$.

Sol: $$V = W - x = \begin{pmatrix} 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

and define the projection matrix.

$$P = \frac{vv^T}{v^Tv} = \frac{1}{20}\begin{bmatrix} 4 & -8 \\ -8 & 16 \end{bmatrix} = \begin{bmatrix} 0.2 & -0.4 \\ -0.4 & 0.8 \end{bmatrix}.$$

Then.
$$H = I - 2P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & -0.8 \\ -0.8 & 1.6 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}.$$

Check that H moves x to w and vice versa.

$$Hx = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} = w.$$

and
$$Hw = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = x.$$

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Q3 (a) Find the Newton's method formula for the following equation.

$$x^3 + x - 1 = 0.$$

Sol:-

Since $f'(x) = 3x^2 + 1$, the formula is given by

$$x_{i+1} = x_i - \frac{x_i^3 + x_i - 1}{3x_i^2 + 1}.$$

$$= 2x_i^3 + 1 / 3x_i^2 + 1.$$

Iterating this formula from initial guess $x_0 = -0.7$ yields

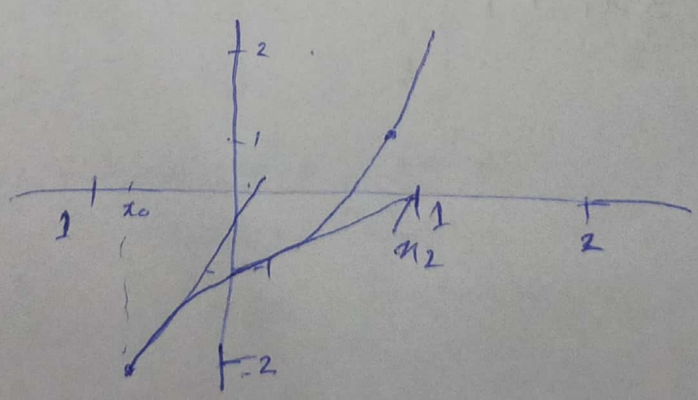
$$x_1 = 2x_0^3 + 1 / 3x_0^2 + 1 = \frac{2(0.7)^3 + 1}{3(-0.7)^2 + 1} \approx 0.1271$$

$$x_2 = \frac{2x_1^3 + 1}{3x_1^2 + 1} \approx 0.9577.$$

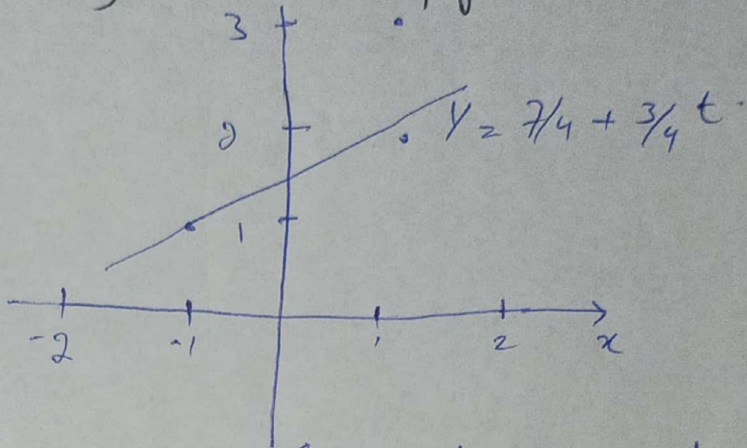
These steps are shown geometrically in fig 1.9 further steps are given in the following table.

i	x_i	$e_i = \lvert x_i - r \rvert$	e_i / e_{i-1}^2
0	-0.70000000	1.38232780	
1	0.12712551	0.55520230	0.2906
2	0.95767812	0.27535032	0.2933
3	0.73482779	0.05249999	0.6924
4	0.68459177	0.00226397	0.8214
5	0.68233217	0.00000437	0.8527
6	0.68232780	0.0000000	0.85541
7	0.68232780	0.0000000	

After only six steps, the root is known to eight correct digits. There is a bit more we can say about the error and how fast it becomes small.



Q3(b) Find the best line that fits the three data points $(t, y) = (1, 2), (-1, 1)$ and $(1, 3)$ in figure below.



Sol: The model is $y = c_1 + c_2 t$ and the goal is to find the best c_1 and c_2. Substitutions of data points into model yields

$$c_1 + c_2(1) = 2.$$
$$c_1 + c_2(-1) = 1.$$
$$c_1 + c_2(1) = 3.$$

or in matrix form,

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

This system has no solution (c_1, c_2) for two separate reason The points are not collinear the equation are inconsistent. So the best solution in term of least square is $(c_1, c_2) = (7/4, 3/4)$. Therefore the best line is

$$y = 7/4 + 3/4\ t$$