Differential Equation

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Question No: 1

a) Define differential equation along with 2 examples?

Differential Equation :-

In mathematics, a differential Equation is an equation that relates one or more functions and represent physical quantities, the derivatives represent their rates of change and the ~~derivatives~~ ~~represent~~ differential equation defines a relationship between the two.

Examples:

1) $U_{xx} + U_{yy} + 0$

2) $\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^2 + 4xy = 4e^{x} \cos x$

Orders of Differential Equation:

(i) First Order Differential Equation

(ii) Second Order Differential Equation

Q1 (b)

Define a Separable Differential Equation (DE)?

Separable Differential Equation :-

Any ODE that can be manipulated into the form where all x's and dx's are one one side of the equation. While all y's and dy's are on the other is called a separable differential equation and can be integrated directly.

Examples :-

(i) $\frac{dy}{dx} = x^2 (y+1) \Rightarrow \frac{1}{y+1} dy = x^2 dx$

(ii) $\frac{dy}{dx} = (y+1) \Rightarrow \frac{1}{y+1} dy = dx$

(i) Solve the following inital value Problem (IVP) using separable D.E and find the interval of validity of the solution.

(a) $y' = \frac{xy^3}{\sqrt{1+x^2}} \quad y(0) = -1$

Solution :-

$$\int \frac{dy}{y^3} = \int \frac{x}{\sqrt{1+x^2}} dx$$

$$\int y^{-3} dy = \int \frac{x}{\sqrt{1+x^2}} dx$$

$$1 + x^2 = u$$

$$2x\,dx = du$$

$$x\,dx = \frac{du}{2}$$

$$\Rightarrow \int y^{-3}\, dy = \int \frac{1}{\sqrt{u}} \frac{du}{2}$$

$$\Rightarrow \frac{y^{-3+1}}{-3+1} = \frac{1}{2} \frac{u^{-1/2+1}}{-1/2+1} + c$$

$$\Rightarrow \frac{y^{-2}}{-2} = \frac{1}{2} \frac{u^{1/2}}{1/2} + c$$

$$\Rightarrow \frac{y^{-2}}{2} = \frac{2}{2} \sqrt{u} + c$$

$$\Rightarrow \frac{1}{-2y^2} = \sqrt{u} + c$$

$$\Rightarrow \frac{1}{-2y^2} = \sqrt{1+x} + c$$

$$\Rightarrow y(0) = -1$$

$$\Rightarrow \frac{1}{-2(-1)^2} = \sqrt{1} + c$$

$$\Rightarrow \frac{1}{-2} = 1 + c$$

$$\Rightarrow -1 - \frac{1}{2} = c$$

$$\Rightarrow c = \frac{-2-y}{2}$$

$$\Rightarrow c = -\frac{3}{2}$$

$$\Rightarrow \frac{y^{-2}}{2} = \sqrt{1+x^2} - 3/2$$

Answer

(b) $y' = e^{-y}(2x-4) \quad y(5)=0$

Solution:-

$$\frac{dy}{dx} = e^{-y}(2x-4)$$

$$\Rightarrow \int \frac{dy}{e^{-y}} = \int (2x-4)\,dx$$

$$\Rightarrow \int e^{y}\,dy = \frac{2x^2}{2} - 4x + C$$

$$\Rightarrow e^{y} = x^2 - 4x + C$$

$$\Rightarrow y = \ln(x^2 - 4x) + C$$

$$\Rightarrow 0 = \ln((5)^2 - 4(5)) + C$$

$$\Rightarrow 0 = \ln(25 - 20) + C$$

$$\Rightarrow 0 = \ln 5 + C$$

$$\Rightarrow C = -\ln 5$$

$$\Rightarrow y = \ln(x^2 - 4x) - \ln 5$$

Answer

Question No : 2

a) Solve the following IVP using Linear Differential method.

(i) Explain the steps for solving Linear Differential Equation.

Steps for solving Linear Differential Equation are as follows.

1) Subsitute $y = uv$ and

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

2) Factor the parts involving v.

3) Put the v term equal to zero (this gives a differential Equation in u and x which can be solved in the next step).

4) Solve using seperation of variable to find u.

5) Subsitute u back into the equation we got at step 2.

6) Solve that to find v.

7) Finally subsitute u and v into $y = uv$ to get solution.

(ii) $\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1 \quad y\left[\frac{\pi}{4}\right] = 3\sqrt{2} \quad 0 \le x \le \frac{\pi}{2}$

Solution:-

Dividing By $\cos x$

$$\Rightarrow y' + \frac{\sin(x)y}{\cos(x)} = \frac{2\cos^2(x)\sin(x) - 1}{\cos(x)}$$

$$\Rightarrow y' + \tan(x)y = \frac{2\cos^3(x).\sin(x) - 1}{\cos(x)} \quad \text{——} (1)$$

It has the form $y' + P(x)y = Q(x)$

where $P(x) = \tan x$ and $Q(x) = \dfrac{2\cos^3 x \sin x - 1}{\cos x}$

Integrating factor $= e^{\int P(x)dx} \cdot e^{\int \tan x dx} \cdot e^{\ln(\sec x)}$

Integrating factor $= \sec x$

Now Xing 1 by I.F or $\sec(x)$

$$\sec x y' + \sec x \tan(x) y = \frac{2\cos^3 x \sin x - 1}{\cos^2 x}$$

$$\frac{d}{dx}[y \sec x] = \frac{2\cos^3 x \sin x - 1}{\cos^2 x}$$

$$\int d[y\sec x] = \int 2\frac{\cos^2(x)\sin(x) - 1}{\cos^2 x}dx$$

By solving the integration we get

$$y \sec x = \frac{-1}{\tan^2 x + 1} - \tan x + c$$

$$y = \cos x\left[-\frac{1}{\tan^2 x+1} - \tan x + c\right]$$

Now

$$y(\pi/4) = 3\sqrt{2}$$

$$\Rightarrow y(\pi/4) = \cos \pi/4\left[\frac{-1}{\tan^2(\pi/4)+1} - \tan(\pi/4) + c\right]$$

$$\Rightarrow 3\sqrt{2} = \frac{1}{\sqrt{2}}\left[-\frac{1}{2} - 1 + c\right]$$

$$\Rightarrow 3 \times 2 = \frac{-3}{2} + c$$

$$\Rightarrow c = -4$$

$$y = \cos x\left[-\frac{1}{\tan^2 x+1} - \tan x - 4\right], 0 \le x \le \pi/2$$

Answer.

Q 2 (iii)

$x' + 2x = \sin t$

Solution:-

$x' + 2x = \sin t \quad -(i)$

This differential Equation has the form $x' + P(t)x = Q(t)$

Where $P(t) = 2$ and $Q(t) = \sin t$

Integrating factor $= e^{\int 2dt} = e^{2t}$

Multiplying (1) by e^{2t} we get

$$e^{2t}x' + 2xe^{2t} = e^{2t}\sin t$$

$$\Rightarrow \frac{d}{dt}\left[xe^{2t}\right] = \int e^{2t}\sin t$$

$$\Rightarrow \int \frac{d}{dt}\left[xe^{2t}\right] = \int e^{2t}\sin t \, dt$$

$$\Rightarrow xe^{2t} = \int e^{2t}\sin t \, dt$$

Solving by integration by parts method

$$\Rightarrow xe^{2t} = \frac{(2\sin t - \cos t)e^{2t}}{5} + C$$

$$\Rightarrow x = e^{-2t}\left[C + \frac{2e^{2t}\sin t}{5} - \frac{e^{2t}\cos t}{5}\right]$$

Answer

Question No : 3

Solve the following IVP for the exact equation and find the interval of validity for the solution.

(i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$, $y(0) = -3$

Solution:-

$$(2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy = 0 \quad \text{——} (*)$$

which is in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Now

$$M = 2xy - 9x^2$$

$$N = 2y + x^2 + 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\, 2xy - 9x^2, \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2y + x^2 + 1)$$

$$\frac{\partial M}{\partial y} = 2x - 0, \quad \frac{\partial N}{\partial x} = 0 + 2x + 0$$

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

So the given Equation (*) is exact

To find general Solution we consider or we check the following conditions

1) $I_1 = \int\limits_{y\ \text{Constant}} M\,dx$

$$I_1 = \int\limits_{y\ \text{Constant}} (2xy - 9x^2)\,dx$$

$$I_1 = 2y\int x\,dx - 9\int x^2\,dx$$

$$I_1 = \frac{\cancel{2}yx^2}{\cancel{2}} - \frac{\overset{3}{\cancel{9}}x^3}{\cancel{3}}$$

$$I_1 = x^2y - 3x^3$$

② $I_2 = \int N$ (terms free form x) dy

$$I_2 = \int (2y + x^2 + 1)\,dy$$

$$I_2 = \int (2y + 1)\,dy$$

$$I_2 = \frac{\cancel{2}y^2}{\cancel{2}} + y$$

$$I_2 = y^2 + y$$

Complete Solution / General Solution

$$I_1 + I_2 = c$$

Putting values of I_1 and I_2 in general solution

$$x^2y - 3x^3 + y^2 + y = c$$

Given Condition:

$$y(0) = -3$$

$y = -3$ and $x = 0$

Put in general solution

$$x^2y - 3x^3 + y^2 + y = c$$

$$(0)(-3) - 3(0) + (-3)^2 + (-3) = c$$

$$9 - 3 = c$$

$$6 = c$$

Put the value of C in general Solution

$$x^2y - 3x^3 + y^2 + y = 6$$

is the required particular Solution of IVP.

ii) $\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0 \quad y(5) = 0$

Solution:-

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))\frac{dy}{dx} = 0$$

$$\left(\frac{2ty}{t^2+1} - 2t\right)dx - (2 - \ln(t^2+1))\,dy = 0$$

$$M(t,y) = \frac{2ty}{t^2+1} - 2t$$

$$N(t,y) = \ln(t^2+1) - 2$$

$$\frac{\partial M}{\partial y} = \frac{2t}{t^2+1}$$

$$\frac{\partial N}{\partial t} = \frac{2t}{t^2+1}$$

$\int M dx + \int (\text{term of } N \text{ free of } x\ dx)$

$$\int\left(\frac{2ty}{t^2+1} - 2t\right)dt + \int -2dy = c$$

$$y\ln(t^2+1) - t^2 - 2y = C$$

$$y(5) = 0$$

$$-(5)^2 = C$$

$$-25 = C$$

Answer