Question $\neq$ 1

The cauchy Euler Equation:-

1 $x^3 y''' + 2x^2 y' + 2y - 10x + \frac{10}{x}$

Solution:-

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{dy}{dx} + 2y = 10x + 10x^{-1}$$

$$x^3 D^3 y + 2x^2 D^2 + 2y = 10x + 10x^{-1}$$

$$(x^3 D^3 + 2x^2 D + 2) y = 10x + 10x^{-1} \quad (1)$$

Let $x = et \implies t = \ln x$

$$xD = D$$

$$x^2 D^2 = D(\Delta - 1) = (\Delta^2 - \Delta)$$

$$x^3 D^3 = \Delta (D - 1)(\Delta - 2)$$

Subtituting into eq (1)

$$(D - 3\Delta^2 + 2D + 2(D^2 - \Delta) + 2) y = 10x + 10x^{-1}$$

$$(\Delta^3 - \Delta^2 + 2) y = 10x + 10x^{-1}$$

$$(m^3 - m^2 + 2)y = 10e^t + \frac{10}{et}$$

Using synthetic division

$$\begin{array}{r|rrrr} & 1 & -1 & 0 & 2 \\ -1 & & & & \\ & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & \underline{\llcorner 0} \end{array}$$

$$D^2 - 2\Delta + 2 = 0$$

Now using Quadratic formula

$$a = 1, \quad b = -2, \quad c = 2$$

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-(-2) \pm \sqrt{-2^2 - 4(1)(2)}}{2a}$$

$$D = \frac{2 \pm \sqrt{4-8}}{2}$$

$$D = \frac{2 \pm \sqrt{-4}}{2}$$

$$D = \frac{2 \pm \sqrt{-4}}{2}$$

$$D = \frac{2 \pm \sqrt{-1} \times \sqrt{4}}{2}$$

$$D = \frac{2 + 2i}{2}$$

$$D = \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

$$D = 1 \pm i$$

Since roots are complen

$$y_c = e^{-x}(c_1 \cos t + c_2 \sin t).$$

Now particular integration

$$y_P = \frac{1}{D^2 - D^2 + 2} \cdot 10e^t + \frac{1}{D^2 - D^2 + 2} \cdot 10/e^t$$

$$= \frac{10et}{(1)^3-(1)^2+2} + \frac{10e^{-t}}{(1)^3-(1)^2+2}$$

$$= \frac{\overset{5}{\cancel{10}}et}{\cancel{2}} + \frac{\overset{5}{\cancel{10}}e^{-t}}{\cancel{2}}$$

$$= 5et + 5et$$

$$yp = 5et + 5et$$

General Solution:-

$$y = yc + yp$$

$$y = e^{-x}(c_1\cos t + c_2\sin t) + 5et + 5e^{-t}$$

put $e^t = x$ and $t = \ln x$

$$y = e^{-x}(c_1 \cancel{t}nx + C_2 \sin\ln x) + 5e^x + 5e$$

Question ≠ 02

2) $x^3\frac{d^3y}{dx} + 4x^2\frac{d^2y}{dx} - 5x\frac{dy}{dx} - 15y = x^4$

Solution:

Let $\frac{d}{dx} = D$

$x^3D^3y + 4x^2D^2y - 5xDy - 15y = x^4$

$(x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4$

Let $x = e^t \Rightarrow t = \ln x.$

$xD = D$

$x^2D^2 = \Delta(\Delta-1) = D^2 - \Delta$

$x^3D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$

Now Subtituting

$(x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4$

$(\Delta^3 - 3\Delta^2 + 2\Delta + 4(\Delta^2-\Delta) - 5(\Delta) - 15)y = e^{4t}$

$(\Delta^3 + \Delta^2 - 7\Delta - 15)y = e^{4t}$

Synthetic division

$$\begin{array}{c|cccc} & 1 & +1 & -7 & -15 \\ 5 & & & & \\ & & 3 & 12 & 15 \\ \hline & 1 & 4 & 5 & \underline{\lfloor 0} \end{array}$$

$\Delta^2 = 4\Delta + 5 = 0$

Quadratic Formula

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2}$$

$$= \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$D = \frac{\cancel{2}(-2 \pm i)}{\cancel{2}}$$

$$y_c = e^{3n}(c_1 \cos t + c_2 \sin t)$$

for $y_p = ?$

$$y_p = \frac{1}{D^3 + D^2 - 7D - 15} e^{4t}$$

$$= \frac{1}{(4)^3 + (4)^2 - 7(4) - 15} e^{4t}$$

$$= \frac{1}{64 + 16 - 28 - 15} e^{4t}$$

$$= \frac{1}{80 - 43} e^{4t}$$

$$y_p = \frac{1}{37} e^{4t}.$$

Hence

$$y = y_c + y_p$$

$$y = (c_1 \cos t + c_2 \sin t) + \frac{1}{37} e^{4t}$$

again put $t = \ln n$ and $n = \ln n$

$$y = e^{3n}(c_1 \cos \ln n + c_2 \sin \log n) + \frac{1}{37} e^{4n} \rightarrow \text{Ans.}$$

Question $\neq$ 03

$$x^2y''' + 2xy' - 6y = 10x^2$$

Solution :-

$$y(1) = 1 \text{ and } y'(1) = -6$$

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 6y = 10x^2$$

$$\Rightarrow (x^2d^2/dx^2 + 2xd/dx - 6)y = 10x^2$$

put $xD = \Delta \Rightarrow x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

$$x = et \text{ and } \log x = t$$

$$(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}$$

$$(\Delta^2 + \Delta - 6)y = 10e^{2t}$$

The characteristic equation

$$\Delta^2 + \Delta - 6 = 0$$

$$\Delta^2 + 3\Delta - 2\Delta - 6 = 0$$

$\Rightarrow D(D+3) - 2(D+3) = 0$

$\Rightarrow (D+3)(D-2) = 0$

$D+3=0 \quad , \quad D-2=0$

$D=2 \quad , \quad D=-3$

Since roots are real and distinct

for $y_c = ?$

$y_c = C_1 e^{-3t} + C_2 e^{2t}$

for $y_p = ?$

$$y_p = \frac{1}{D^2-D-6} 10^{2t}$$

$$= \frac{10}{D^2-D-6} e^{2t}$$

$$= 10 \frac{1}{0} e^{2t} \quad \text{fails}$$

Now

$$10 \frac{1}{d/dD\,(D^2+D-6)} e^{2t}$$

$$\Rightarrow 10 \frac{t}{2D+1} e^{2t}$$

$$= 10 \frac{1^{-t}}{4+1} e^{2t}$$

$$y_p = 2te^{2t}.$$

General Solution:

$$y = y_c + y_p$$

$$= C_1 e^{-3t} + C_2 e^{2t} + 2te^{2t}$$

$$y = C_1 x^{-3} + C_2 x^2 + 2(\log x)\, x^2 \text{ --- } (B)$$

put $y(1) = 1$ i-e $x = 1$, $y = 1$ in (B)

$$1 = C_1 (1)^{-3} + C_2 (1)^2 + 2\log(1)$$

$$1 = C_1 + C_2 \longrightarrow (C)$$

Now differentiate eq (B) w.r.t x.

$y' = -3c_1 x^{-4} + 2c_2 x + \frac{2}{x}(x^2) + 4x \log$

Now put $y'(1) = -6$ i-e $y' = -6$ and $x = -6$

$-6 = -3c_1 + 2c_2 + 2 + 0$

$\Rightarrow -6 = -3C_1 + 2C_2 + 2$

$\Rightarrow -6-2 = -3C_1 + 2C_2 + 2$

$= -8 = -3C_1 + 2C_2$ — (D)

Xing eq (c) with (2) and -ing from (D)

$$\begin{array}{l} 2C_1 + 2C_2 = 2 \\ \mp 3C_1 \pm 2C_2 = \mp 8 \\ \hline 5C_1 \quad = 10 \end{array}$$

$C_1 = \frac{\cancel{10}^{2}}{\cancel{5}_{1}}$ $\boxed{C_1 = 2}$

$-8 = -3(2) + 2C_2$

$-8 = -6 + 2C_2$

$2C_2 = -8 + 6$

$2C_2 = -2$

$$C_2 = \frac{-\cancel{2}}{\cancel{2}} \; 1$$

$$\boxed{C_2 = -1}$$

Now put the value of C_1 and C_2 in eq (B).

$$y = 2x^{-3} - x^2 + 2 \ln x \; x(x^2)$$

$$y = \frac{2}{x^3} - x^2 + 2x^2 \log x \quad \text{— Ans.}$$

Question # 04

$$x^2y'' + 7xy' + 5y = x^5$$

$$y(0) = 2 \text{ and } y'(1) = 2$$

Solution :

$$x^2\frac{dy^2}{dx^2} + 7x\frac{dy}{dx} + 5y = x^5$$

$$\Rightarrow \left(x^2\frac{d^2}{dx^2} + 7x\frac{d}{dx} + 5\right)y = x^5 \quad \text{— (A)}$$

Put $xD = \Delta \Rightarrow x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

$x = et \Rightarrow \log x = t$ in eq (A)

$$\Rightarrow (\Delta^2 - \Delta + 7\Delta + 5)y = e^{5t}$$

$$\Rightarrow (\Delta^2 + 6\Delta + 5)y = e^{5t}$$

By Quadratic Formula

$$\Delta = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36 - 20}}{2}$$

$$= \frac{-6 \pm \sqrt{16}}{2}$$

$$= \frac{-6 \pm \sqrt{4^2}}{2}$$

$$= \frac{\cancel{2}(-3 \pm 2)}{\cancel{2}}$$

$$D = -3 \pm 2$$

Since roots are real and distinct

$$y_c = C_1 e^{-5t} + C_2 e^{-t}$$

for $y_p = ?$

$$y_p = \frac{1}{D^2 + 6D + 5} e^{5t}$$

$$= \frac{1}{(5)^2 + 6(5) + 5} e^{5t}$$

$$= \frac{1}{60} e^{5t}$$

Now General Solution is

$$y = y_c + y_p$$

$$y = C_1 e^{-5t} + C_2 e^{-t} + \frac{1}{60} e^{5t}$$

$$y = C_1 x^{-5} + C_2 x^{-1} + \frac{1}{60} x^5 \longrightarrow (B)$$

$x = 0$ put in this equation

No in eq (B) $e^0 = 1$

put $y(0) = 2$ i-e $y = 2$ and $x = 2$

$$2 = C_1 (2)^{-5} + C_2 (2)^{-1} + \frac{1}{60} (2)^5$$

$$2 = -32C_1 - 2C_2 + \frac{1}{\cancel{60}\,15} (\cancel{32}\,8)$$

$$2 = -32C_1 - 2C_2 + 8/15$$

$$2 - \frac{18}{15} = -32C_1 - 2C_2$$

$$\frac{22}{15} = -32C_1 - 2C_2 \rightarrow (C)$$

Now differentiate eq (B) w.r.t (x)

$$y' = -5C_1 x^{-6} - C_2 x^{-2} + \frac{1}{12} x^4 \rightarrow$$

put $y'(1) = 2$ i-e $y' = 2$ and $x = 2$ in above equation.

$$2 = -5C_1 C_2^{-6} - C_2 (2)^{-2} + \frac{1}{12} (2)^4$$

$$2 = -5C_1 (-64) - C_2 (4) + \frac{1}{12} (16)$$

$$2 = 320C_1 + 4C_2 + 4/3$$

$$\Rightarrow 2 - \frac{4}{3} = 320C_1 + 4C_2$$

$$\Rightarrow 2/3 = 320C_1 + 4C_2 \rightarrow (D)$$

Xing eq (c) with 2 and then —ing eq (c) from (D).

$$-\frac{44}{15} = 64C_1 + 4C_2$$

$$-\frac{44}{15} = 64C_1 + 4C_1$$

$$\underline{\mp \tfrac{2}{3} = \pm 320C_1 \pm 4C_2}$$

$$\frac{34}{15} = -256C_1$$

$$C_1 = \frac{34}{15} \times 256$$

$$\boxed{C_1 = 580}$$

put the value of C_1 in eq (c)

$$\frac{22}{15} = -32(580) - 2C_2$$

$$\Rightarrow \frac{22}{15} = -18560 - 2C_2$$

$$\Rightarrow \frac{22}{15} + 18560 = -2C_2$$

$$\Rightarrow \frac{18561}{-2} = C_2$$

$$\boxed{-9280 = C_2}$$

Now put the value of C_1 and C_2 in eq (B).

$$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x^5$$

$$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5 \text{ — Ans.}$$