final term

4th Semester

BE(C)

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Section - A

Subject:- Differential Equations

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Question no. 01 (a).

(i)

Sol:- As given;

Symmetry of waves is expressed by one-dimensional wave equation.

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

where

w = wave height
x = distance variable
t = time variable
c = velocity.

Now, to show that;

$$w = \sin(x+ct) + \cos(2x+2ct)$$

is a solution of wave equation

we need to find partial derivative.

$\Rightarrow$ $\cancel{\partial w}$

$\longrightarrow$

$w = \sin(x+ct) + \cos(2x+2ct)$

Applying partial derivative

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial t}\left(\sin(x+ct) + \cos(2x+2ct)\right)$$

$$\frac{\partial w}{\partial t} = \cos(x+ct) + c(-\sin(2x+2ct)) + 2c$$

again applying derivative:

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) + c^2 - \cos(2x+2ct) + 4c^2$$

$\hookrightarrow (i)$

$$\frac{\partial w}{\partial t} = \cos(x+ct) - \sin(2x+2ct) + 2$$

and

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) - \sin(2x+2ct) + 2$$

$$\Rightarrow + c^2\left[-\sin(x+ct) - 4\cos(x+2ct)\right]$$

$\longrightarrow$

$$\Rightarrow \frac{\partial^2 w}{\partial t^2} = +c^2\left[-\sin(x+ct) - 4\cos(2x+2ct)\right]$$

$$= c^2 \cdot \frac{\partial^2 w}{\partial t^2} \quad \text{Ans.}$$

x ———————————————— x

(ii) $w = \tan(2x+ct)$

Sol:-

Given that,

$w = \tan(2x+ct)$

To check if it is the sol. of the given eq. or not,

Diff. partial Diff. w.r.t t

Now

$$\frac{\partial w}{\partial t} = c\sec^2(2x+ct).$$

again Diff.

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(c\sec^2(2x+ct)\right)$$

$$= 2c^2 \cdot 2\sec^2(2x+ct)\tan(2x+ct)$$

$$\Rightarrow \frac{\partial w}{\partial x} = 2\sec^2(2x+ct)$$

and $\frac{\partial^2 w}{\partial x^2} = 4\sec^2(2x+ct)\tan(2x+ct)$

$\longrightarrow$

$$\Rightarrow 4c^2 \sec^2(2x+ct)\tan(2x+ct)$$

$$= 4c^2 \sec^2(2x\cancel{+}ct)\tan(2x+ct)$$

$$\Rightarrow 0 = 0$$

Satisfied;

Hence

$w = \tan(2x+ct)$ is the

sol. of given eq.

Question No. 02

Sol:-

Given that;

$F(x) = x \qquad -\pi < x \leq 0$

$\quad = 2x \qquad 0 \leq x \leq \pi$

Expanding in a fourier Series.

$$f(x) = \begin{cases} x; & -\pi < x \leq 0 \\ 2x; & 0 \leq x \leq \pi \end{cases}$$

we need to find furior Co-efficients, a_0, a_n, b_n

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left(0 - \frac{\pi^2}{2}\right) + \frac{2}{\pi}\left(\frac{\pi^2}{2} - 0\right)$$

$$\Rightarrow a_0 = \frac{-\pi}{2} + \pi = \pi/2$$

$$\Rightarrow \boxed{a_0 = \pi/2} \longrightarrow 1$$

a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x(\cos nx \, dx) + \frac{1}{\pi}\int_{0}^{\pi} 2x\cos nx \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left(\frac{1-(-1)^n + 2(-1)^n - 2}{n^2}\right) = \frac{(-1)^n - 1}{\pi n^2}$$

$$\boxed{a_n = \begin{cases} \frac{-2}{\pi n^2}; & \text{if } n \text{ is odd} \\ 0; & \text{if } n \text{ is even.} \end{cases}} \rightarrow (2)$$

Now

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$\Rightarrow b_n = \frac{1}{\pi}\left(-\frac{\pi\cos n\pi}{n}\right) + \frac{2}{\pi}\left(\frac{-\pi\cos n\pi}{n}\right)$$

$$\Rightarrow b_n = \frac{-3\cos n\pi}{n}$$

$$\Rightarrow \boxed{b_n = \frac{3(-1)^{n+1}}{n}} \rightarrow (3)$$

Hence from eq (1) (2) and (3)

The required fourier series is

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{2n-1} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

Question - 03

Solve the initial value problem:

$$y'' - 4y' + 13y = 8 \sin 3x, \quad y(0) = 1, \quad y'(0) = 2$$

Sol:- Given that:

$$y'' - 4y' + 13y = 8 \sin 3x \rightarrow (i)$$

(Homogeneous equation)

Change eq(i) into auxillary eq.

$$m^2 - 4m + 13 = 0 \rightarrow (ii)$$

Solve eq(ii) By Quadratic formula;

$$a = 1, \quad b = -4, \quad c = 13$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$\longrightarrow$

$$\Rightarrow m = \frac{4 \pm \sqrt{-36}}{2}$$

$$m = \frac{4 \pm \sqrt{36i}}{2}$$

$$m = \frac{4 \pm 6i}{2} \Rightarrow m = 2 \pm 3i$$

$$\Rightarrow m_1 = 2 + 3i$$
$$m_2 = 2 - 3i$$

$$\Rightarrow y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) \rightarrow (ii)$$

Let $y_p = A \cos 3x + B \sin 3x \longrightarrow (iii)$

Diff w.r.t 'x'

$$y_p' = -3A \sin 3x + 3B \cos 3x$$

Again Diff eq(iii) w.r.t x'

$$y_p'' = -9A \cos 3x - 9B \sin 3x$$

put in(i)

$$\Rightarrow -9A \cos 3x - 12B \cos 3x + 13A + 13A \cos 3x - 9B \sin 3x$$
$$+ 12A \sin 3x + 13B \sin 3x - 8 \sin 3x$$

$\longrightarrow$

$= (4A - 12B)\cos 3x + (4B + 12A)\sin 3x = 8\sin 3x$

By Comparing Co-efficients.

$\underline{\sin 3x} \Rightarrow$

$4B + 12A = 8 \longrightarrow (1)$

$4A - 12B = 0$

$\Rightarrow 4A = 12B$

$\Rightarrow \boxed{A = 3B} \longrightarrow (2)$

put eq (2) in eq (1)

$4B + 12(3B) = 8$

$40B = 8$

$\Rightarrow \boxed{B = 1/5} \longrightarrow (3)$

put value of 'B' in eq (2)

$A = 3(1/5)$

$\Rightarrow \boxed{A = 3/5} \longrightarrow (4)$

put eq (3), (4) in eq (iii)

$y_p = \cancel{B}\ 3$

$y_p = \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \rightarrow (A)$

General Solution is

$y = y_c + y_p$

$y = e^{2x}\left(c_1 \cos 3x + c_2 \sin 3x\right) + \frac{3}{5}\cos 3x$

$+ \frac{1}{5}\sin 3x) \rightarrow (B)$

Now, to find values of 'c_1' and 'c_2'

Given that $y(0) = 1$

put $x = 0$, $y = 1$ in eq (B)

$1 = e^{2(0)}\left(c_1 \cos 3(0) + c_2 \sin 3(0)\right) + \frac{3}{5}\cos 3(0)$

$+ \frac{1}{5}(\sin 3(0)))$

$1 = c_1(1) + c_2(0) + \frac{3}{5}(1) + \frac{1}{5}(0)$

$1 = c_1 + 3/5$

$c_1 = 2/5 \rightarrow (55)$

$\rightarrow$

$\Rightarrow$ Diffi eq(2) w.r.t x

$$y' = C_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + C_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x)$$

$$\frac{6}{5}\sin 3x + \frac{3}{5}\cos 3x \longrightarrow (D)$$

put $y' = 2$, $x = 0$ in D

$\therefore y'(0) = 2$

$$y' = C_1\left(2e^{2x}\cos 3x - 3e^{2x}\sin 3x\right) + C_2\left(2e^{2x}\sin 3x + 3e^{2x}\cos 3x\right)$$

$$\Rightarrow 2 = C_1\left(2e^{2(0)}\cos 3(0) - 3e^{2(0)}\sin 3(0)\right) + C_2\left(2e\sin 3(0) + 3e^{2(0)}(\cos(3)(0))\right) - \frac{6}{5}\sin 3(0) + \frac{3}{5}\cos 3(0)$$

$$2 = C_1(2) + C_2(3) - 0 + \frac{3}{5}$$

$$2 = 2C_1 + 3C_2 + \frac{3}{5}$$

put $C_1 = 2/5$

$$2 = 4/5 + 3C_2 + \frac{3}{5}$$

$$2 = \frac{7}{5} + 3C_2$$

$$3C_2 = \frac{3}{5} \Rightarrow C_2 = 3/15 \longrightarrow (60)$$

put values of c_1 and c_2 in eq(60)

$$\Rightarrow y = \frac{2}{5} e^{2x} \cos 3x + \frac{3}{15} e^{2x} \sin 3x + \frac{3}{5} \cos 3x + \frac{1}{3} \sin 3x.$$

Required General solution.

Question No. 04

Solve.

$$(D^2 - DD') = \cos x \cos 2y.$$

Sol:

Given that;

$$(D^2 - DD') = \cos x \cdot \cos 2y \rightarrow (i)$$

we know that;

(Auxillary equation) $\frac{D}{D'} = m \Rightarrow D = m, D' = 1$

$\Rightarrow D^2 - DD' = 0$

$\Rightarrow m^2 - m = 0$

$\Rightarrow m = 0, 1$

Therefore;

(Complementary function) $C.F = f_1(y) + f_2(y+x)$

from eq (i)

$$P.I = \frac{1}{D^2 - DD'} \cos x \cdot \cos 2y$$

$$= \frac{1}{2}\frac{1}{D^2 - DD'} 2\cos x \cdot \cos 2y$$

∴ multiply and divide by '2'.

As;

$$2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$$

$$\Rightarrow C.F = F_1(y-x) + xf_2(y-x)$$

$$P.I = \frac{1}{D^2+2DD'+D'^2}\left[2(y-x) + \sin(x-y)\right]$$

$$= \frac{1}{(D+D')^2}\left(2(y-x) + \sin(x-y)\right)$$

By General method;

$$m = -1, \quad y - x = C$$

$$= \frac{1}{D+D'}\left(2C + \sin(-C)\right)dx$$

$$= \frac{1}{D+D'}\left(2Cx - \sin(x)\right)$$

→

Replace 'c' by $(y-x)$

$$= \frac{1}{D+D'}\left(2x(y-x) - x\sin(y-x)\right)$$

Again replace 'y-x' by 'c'

$$= \int (2xc - x\sin c)\,dx$$

$$\Rightarrow cx^2 - \frac{x^2}{2}\sin c$$

again replace 'c' by 'y-x'

$$= x^2(y-x) - \frac{x^2}{2}\sin(y-x)$$

$$= x^2y - x^3 + \frac{x^2}{2}\sin(x-y)$$

The required sol:

$$\boxed{z = C.F + P.I = f_1(y-x) + x f_2(y-x) + x^2y - x^3 + \frac{1}{2}x^2\sin(x-y)}$$

Ans.