

final term

BE (C)

4th Semester

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Section - A

Subject: Differential Equations

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Question no. 01 (a)

Sol: (i) As given:

Symmetry of waves is expressed by one-dimensional wave equation.

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

Where

$w$  = wave height

$x$  = distance variable

$t$  = time variable

$c$  = velocity.

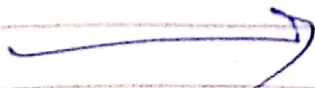
Now, to show that:

$$w = \sin(x+ct) + \cos(2x+2ct)$$

is a solution of wave equation

we need to find partial derivative.

⇒  $\frac{\partial w}{\partial x}$



(2)

$$w = \sin(x+ct) + \cos(2x+2ct)$$

Applying partial derivative

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial t} (\sin(x+ct) + \cos(2x+2ct))$$

$$\frac{\partial w}{\partial t} = \cos(x+ct) + c - \sin(2x+2ct) + 2c$$

again applying derivative:

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) + c^2 - \cos(2x+2ct) + 4c^2$$

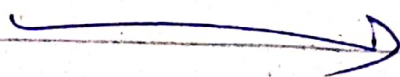
↳ (i)

$$\frac{\partial w}{\partial t} = \cos(x+ct) - \sin(2x+2ct) + 2$$

and

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) - \sin(2x+2ct) + 2$$

$$\Rightarrow +c^2 \left[ -\sin(x+ct) - 4\cos(2x+2ct) \right]$$



(3)

$$\Rightarrow \frac{\partial^2 w}{\partial t^2} = -c^2 \left( -\sin(x+ct) - 4c \cos(2x+2ct) \right)$$

$$= c^2 \frac{\partial^2 w}{\partial t^2} \quad \text{Ans.}$$

+

x

(4)

$$(ii) w = \tan(2x + ct)$$

Sol:

Given that,

$$w = \tan(2x + ct)$$

To check if it is the sol. of the given eq. or not,

Diff: partial Diff. w.r.t ~~x~~

Now

$$\frac{\partial w}{\partial t} = c \sec^2(2x + ct)$$

again Diff:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t} (c \sec^2(2x + ct))$$

$$= 2c^2 \sec^2(2x + ct) \tan(2x + ct)$$

$$\Rightarrow \frac{\partial w}{\partial x} = 2 \sec^2(2x + ct)$$

$$\text{and } \frac{\partial^2 w}{\partial x^2} = 4 \sec^2(2x + ct) \tan(2x + ct)$$



$$\Rightarrow 4c^2 \sec^2(2x+ct) \tan(2x+ct)$$

$$= 4c^2 \sec^2(2x+ct) \tan(2x+ct)$$

$$\Rightarrow 0 = 0$$

Satisfied;

Hence

$w = \tan(2x+ct)$  is the  
sol. of given eq.

x

## Question No. 02

Sol:-

Given that:

$$f(x) = \begin{cases} x & -\pi < x \leq 0 \\ 2x & 0 \leq x \leq \pi \end{cases}$$

Expanding in a fourier series.

$$f(x) = \begin{cases} x; & -\pi < x \leq 0 \\ 2x; & 0 \leq x \leq \pi \end{cases}$$

We need to find further  
co-efficients,  $a_0, a_n, b_n$ 

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 x dx + \frac{1}{\pi} \int_0^{\pi} 2x dx$$

$$= \frac{1}{\pi} \left( \frac{x^2}{2} \right)_{-\pi}^0 + \frac{2}{\pi} \left( \frac{x^2}{2} \right)_0^{\pi}$$

$$= \frac{1}{\pi} \left( 0 - \frac{\pi^2}{2} \right) + \frac{2}{\pi} \left( \frac{\pi^2}{2} - 0 \right)$$

$$\Rightarrow a_0 = \frac{-\pi + \pi}{2} = \frac{\pi}{2}$$

$$\Rightarrow \boxed{a_0 = \pi/2} \rightarrow \text{①}$$

Ans:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 x \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} 2x \cos nx \, dx$$

$$= \frac{1}{\pi} \left[ x \left( \frac{\sin nx}{n} \right) - \left( -\frac{\cos nx}{n^2} \right) \right]_{-\pi}^0$$

$$+ \frac{2}{\pi} \left[ x \left( \frac{\sin nx}{n} \right) - \left( -\frac{\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi} \left[ \frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2} \right] + \frac{2}{\pi} \left[ \frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2} \right]$$

$$= \frac{1}{\pi} \left[ \frac{1 - (-1)^n + 2(-1)^n - 2}{n^2} \right] = \frac{(-1)^n - 1}{\pi n^2}$$

$$a_n = \begin{cases} \frac{-2}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \rightarrow \textcircled{2}$$



Now

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^0 x \sin nx \, dx + \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi} \left[ x \left( -\frac{\cos nx}{n} \right) - \left( -\frac{\sin nx}{n^2} \right) \right]_{-\pi}^0 + \frac{2}{\pi} \left[ x \left( -\frac{\cos nx}{n} \right) - \left( -\frac{\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$\Rightarrow b_n = \frac{1}{\pi} \left( -\frac{\pi \cos n\pi}{n} \right) + \frac{2}{\pi} \left( \frac{-\pi \cos n\pi}{n} \right)$$

$$\Rightarrow b_n = -\frac{3 \cos n\pi}{n}$$

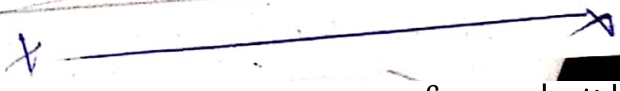
$$\Rightarrow b_n = \frac{3(-1)^{n+1}}{n} \rightarrow \text{③}$$

Hence from eq ① ② and ③

The required Fourier series is

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{2n-1} + 3 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$$



## Question - 03

Solve the initial value problem:

$$y'' - 4y' + 13y = 8 \sin 3x, \quad y(0) = 1, \quad y'(0) = 2$$

Sol:- Given that:

$$y'' - 4y' + 13y = 8 \sin 3x \rightarrow (i)$$

(Homogeneous equation)

change eq (i) into auxiliary eq.

$$m^2 - 4m + 13 = 0 \rightarrow (ii)$$

Solve eq (ii) By Quadratic formula,

$$a = 1, \quad b = -4, \quad c = 13$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

2



$$\Rightarrow m = \frac{4 \pm \sqrt{-36}}{2}$$

$$m = \frac{4 \pm \sqrt{36}i}{2}$$

$$m = 2 \pm 3i$$

$$\Rightarrow m = 2 \pm 3i$$

$$\Rightarrow m_1 = 2 + 3i$$

$$m_2 = 2 - 3i$$

$$\Rightarrow y_c = e^{2x} (C_1 \cos 3x + C_2 \sin 3x) \rightarrow (ii)$$

$$\text{Let } y_p = A \cos 3x + B \sin 3x \rightarrow (iii)$$

Diff w.r.t 'x'

$$y_p' = -3A \sin 3x + 3B \cos 3x$$

Again Diff eq(iii) w.r.t 'x'

$$y_p'' = -9A \cos 3x - 9B \sin 3x$$

put in (i)

$$\Rightarrow -9A \cos 3x - 12B \cos 3x + 12A + 12A \cos 3x - 9B \sin 3x + 12A \sin 3x + 12B \sin 3x - 8 \sin 3x$$

$$= (4A - 12B) \cos 3x + (4B + 12A) \sin 3x = 8 \sin 3x$$

By Comparing co-efficients.

$$\underline{\underline{\sin 3x}} \Rightarrow$$

$$4B + 12A = 8 \rightarrow (1)$$

$$4A - 12B = 0$$

$$\Rightarrow 4A = 12B$$

$$\Rightarrow \boxed{A = 3B} \rightarrow (2)$$

Put eq (2) in eq (1)

$$4B + 12(3B) = 8$$

$$40B = 8$$

$$\Rightarrow \boxed{B = \frac{1}{5}} \rightarrow (3)$$

Put value of 'B' in eq (2)

$$A = 3\left(\frac{1}{5}\right)$$

$$\Rightarrow \boxed{A = \frac{3}{5}} \rightarrow (4)$$

Put eq (3), (4) in eq (iii)

$$y_p = \frac{3}{5}$$



$$y_p = \frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x \rightarrow (A)$$

General Solution is

$$y = y_c + y_p$$

$$y = e^{2x} \left( C_1 \cos 3x + C_2 \sin 3x + \frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x \right) \rightarrow (B)$$

Now, to find values of 'C<sub>1</sub>' and 'C<sub>2</sub>'

Given that  $y(0) = 1$

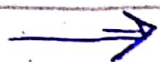
Put  $x=0$ ,  $y=1$  in eq(B)

$$1 = e^{2(0)} \left( C_1 (\cos 3(0)) + C_2 (\sin 3(0)) + \frac{3}{5} (\cos 3(0)) + \frac{1}{5} (\sin 3(0)) \right)$$

$$1 = C_1 (1) + C_2 (0) + \frac{3}{5} (1) + \frac{1}{5} (0)$$

$$1 = C_1 + \frac{3}{5}$$

$$C_1 = \frac{2}{5} \rightarrow (55)$$



⇒ Diff. eq (2) w.r.t  $x$

$$y' = C_1(2e^{2x} \cos 3x - 3e^{2x} \sin 3x) + C_2(2e^{2x} \sin 3x + 3e^{2x} \cos 3x)$$

$$\frac{6}{5} \sin 3x + \frac{3}{5} \cos 3x \rightarrow \textcircled{D}$$

put  $y' = 2$ ,  $x = 0$  in D

$$\therefore y'(0) = 2$$

$$y' = C_1(2e^{2x} \cos 3x - 3e^{2x} \sin 3x) + C_2(2e^{2x} \sin 3x + 3e^{2x} \cos 3x)$$

$$\Rightarrow 2 = C_1(2e^{2(0)}(\cos 3(0)) - 3e^{2(0)} \sin 3(0)) + C_2(2e^{2(0)} \sin 3(0) + 3e^{2(0)}(\cos 3(0)))$$

$$- \frac{6}{5} \sin 3(0) + \frac{3}{5} \cos 3(0)$$

$$2 = C_1(2) + C_2(3) - 0 + \frac{3}{5}$$

$$2 = 2C_1 + 3C_2 + \frac{3}{5}$$

put  $C_1 = \frac{2}{5}$

$$2 = \frac{4}{5} + 3C_2 + \frac{3}{5}$$

$$2 = \frac{7}{5} + 3C_2$$

$$3C_2 = \frac{3}{5} \Rightarrow C_2 = \frac{1}{5}$$

$$C_2 = \frac{3}{15} \rightarrow \textcircled{60}$$

put values of  $C_1$  and  $C_2$   
in eq (60)

if

$$\Rightarrow y = \frac{2}{5} e^{2x} \cos 3x + \frac{3}{15} e^{2x} \sin 3x + \frac{3}{5} \cos 3x + \frac{1}{3} \sin 3x.$$

Required General  
solution.

x ————— x

Question No. 04

Solve.

$$(D^2 - DD') = \cos x \cos 2y$$

Sol:

Given that:

$$(D^2 - DD') = \cos x \cdot \cos 2y \rightarrow (i)$$

We know that:

(Auxiliary equation)  $\frac{D}{D'} = m \Rightarrow D = m, D' = 1$

$$\Rightarrow D^2 - DD' = 0$$

$$\Rightarrow m^2 - m = 0$$

$$\Rightarrow m = 0, 1$$

Therefore:

(Complementary) C.F =  $f_1(y) + f_2(y+x)$   
function

from eq (i)

$$P.O.F = \frac{1}{D^2 - DD'} \cos x \cdot \cos 2y$$





$$= \frac{1}{2} \frac{1}{D^2 - DD'} 2 \cos x \cdot \cos 2y$$

∴ multiply and divide by '2'.

As,

$$2 \cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$$

$$\Rightarrow C.F = f_1(y-x) + x f_2(y-x)$$

$$P.I = \frac{1}{D^2 + 2DD' + D'^2} \left[ 2(y-x) + \sin(x-y) \right]$$

$$= \frac{1}{(D+D')^2} \left[ 2(y-x) + \sin(x-y) \right]$$

By General method:

$$m = -1, y-x = c$$

$$= \frac{1}{D+D'} (2c + \sin(-c)) dx$$

$$= \frac{1}{D+D''} (2cx - \sin(c))$$

→

Replace 'C' by (y-x)

$$= \frac{1}{D+D'} (2x(y-x) - x \sin(y-x))$$

Again replace 'y-x' by 'C'

$$= \int (2xC - x \sin C) dx$$

$$\Rightarrow C^2 x^2 - \frac{x^2}{2} \sin C$$

again replace 'C' by 'y-x'

$$= x^2(y-x) - \frac{x^2}{2} \sin(y-x)$$

$$= x^2 y - x^3 + \frac{x^2}{2} \sin(x-y)$$

The required sol:

$$Z = C.F + P.I = f_1(y-x) + x f_2(y-x) + x^2 y - x^3 + \frac{1}{2} x^2 \sin(x-y)$$

X

Ans.