Iqra National University

Linear Algebra
Final Paper
Name: Nooriya Ilyas
ID : 15770
BS(SE) II

Question 1. Determine if the following system is consistent or not.

$$\begin{aligned} x_1 - 7x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned}$$

Solution:-

Consistent systems of equations means that for a given system, there exists one solution set at least. In other words, as long as we can find a solution for the system of equations, it is referred to as consistent. But if there is not even a single solution for a system of equations, it is called inconsistent system.

Lets find out if the following matrix is consistent or in-consistent.

$$
\begin{aligned}
x_1 - 7x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
5x_1 - 5x_3 &= 10.
\end{aligned}
$$

writing this in augmented matrix form

$$\left[\begin{array}{ccc|c} 1 & -7 & 1 & 0 \\ 0 & 2 & 8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

performing row operations:

$R_3 - 5(R_1) \Rightarrow R_3$:-

$$\left[\begin{array}{ccc|c} 1 & -7 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 35 & -10 & 10 \end{array}\right]$$

$\frac{-35}{2} \times R_2 + R_3 \Rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & -7 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 130 & -150 \end{array}\right]$$

This indicates that $130 = -150$, we know that this is impossible so this system is inconsistent.

Question 2 Find the inverse of

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 1 & 4^{th} ID \\ 5 & -2 & 7 \end{bmatrix}$$ by adjoint method.

4^{th} ID $= 7$

Solution:-

we know the formula:-

$$A^{-1} = \frac{1}{|A|} \text{ adj. } A$$

$$|A| = 3\begin{vmatrix} -1 & 7 \\ -2 & 7 \end{vmatrix} - 4\begin{vmatrix} 2 & 7 \\ 5 & 7 \end{vmatrix} + 5\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix}$$

$$|A| = 3\big((-1\times7)-(-2\times7)\big) - 4\big((2\times7)-(7\times5)\big) + 5\big((-2\times2)-(5\times-1)\big)$$

$$|A| = 21 + 84 + 5 = 110.$$

Now finding adj A.

adj A.

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 7 \\ 5 & -2 & 7 \end{bmatrix}$$

$$a_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 7 \\ -2 & 7 \end{vmatrix} = 7$$

$$a_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 7 \\ 5 & 7 \end{vmatrix} = 21$$

$$a_{13} = (-1)^{1+3} \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = 1$$

$$a_{21} = (-1)^{2+1} \begin{vmatrix} 4 & 5 \\ -2 & 7 \end{vmatrix} = -38$$

$$a_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} = -4$$

$$a_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} = 26$$

$$a_{31} = (-1)^{3+1} \begin{vmatrix} 4 & 5 \\ -1 & 7 \end{vmatrix} = 33$$

$$a_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 5 \\ 2 & 7 \end{vmatrix} = -11$$

$$a_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = -11$$

writing the cofactors inside the matrix and taking transpose.

$$\begin{bmatrix} 7 & 21 & 1 \\ -38 & -4 & 26 \\ 33 & -11 & -11 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 7 & -38 & 33 \\ 21 & -4 & -11 \\ 1 & 26 & -11 \end{bmatrix}$$

$$A^{-1} = \frac{1}{110} \begin{bmatrix} 7 & -38 & 33 \\ 21 & -4 & -11 \\ 1 & 26 & -11 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{7}{110} & \frac{-19}{55} & \frac{3}{10} \\ \frac{21}{110} & \frac{-2}{55} & \frac{-1}{10} \\ \frac{1}{110} & \frac{13}{55} & \frac{-1}{10} \end{bmatrix}$$ Answer.

Question. 3. Solve the following systems of linear equations by Gauss Jordan method.

$$2x + 2y + 4z = 18$$
$$x + 3y + 2z = 13$$
$$3x + 2y \cancel{+} -3z = 14$$

Solution:-

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 18 \\ 1 & 3 & 2 & 13 \\ 3 & 2 & -3 & 14 \end{array}\right]$$

★ $R_1 = \frac{R_1}{R_2}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 1 & 3 & 2 & 13 \\ 3 & 2 & -3 & 14 \end{array}\right]$$

★ $R_2 = R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & 0 & 4 \\ 3 & 2 & -3 & 14 \end{array}\right]$$

★ $R_3 = R_3 - (3R_1)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & 0 & 4 \\ 0 & -1 & -9 & -13 \end{array}\right]$$

★ $R_2 = \frac{R_2}{2}$ (Divide by 2).

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & -9 & -13 \end{array}\right]$$

★ $R_1 = R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & -9 & -13 \end{array}\right]$$

★ $R_3 = R_3 + R_2$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -9 & -11 \end{array}\right]$$

★ $R_3 = \frac{R_3}{-9}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{11}{9} \end{array}\right]$$

★ $R_1 = R_1 - (2R_3)$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{41}{9} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{11}{9} \end{array}\right]$$

Answer.

Question. 4. Show that this matrix is diagonalisable.

$$\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

Solution:-

First we will find the eigenvalues and eigenvectors

Starting by forming a new matrix by subtracting λ from the diagonal elements of the given matrix.

$$\begin{bmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{bmatrix}$$

finding the determinant :

$$= -\lambda^3 + 8\lambda^2 - 17\lambda + 10$$

after solving the equation,
we get $\lambda_1 = 5$, $\lambda_2 = 2$, $\lambda_3 = 1$.
These are the
eigen values.

Next, finding the eigen vectors:-

a). $\lambda = 5$:-

$$\begin{bmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix}$$

after obtaining the reduced
row echelon form,
Solving the matrix equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

if we take $v_3 = t$, then
$v_1 = 0$, $v_2 = t$.

Therefore , $v = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t.$

b) $\lambda = 2$:-

eigen vector:- $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$

c) $\lambda = 1$:-

eigen vector :- $\begin{bmatrix} 1/2 \\ 1/4 \\ 1 \end{bmatrix}$

Now we are going to make a matrix P, which consist of the three eigen vectors.

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/4 \\ 1 & 1 & 1 \end{bmatrix}$$

The D matrix has the eigen values along the diagonal, and zeros in the rest of the matrix

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer.

And these matrices have the property that $A = PDP^{-1}$

Question. 5. Determine if the following homogeneous system has a non-trivial solution. Then describe the solution set.

$$3x_1 + 5x_2 - 4x_3 = 0$$

$$-3x_1 \cancel{+} - 25x_2 + 4x_3 = 0$$

$$6x_1 + x_2 - 8x_3 = 0$$

Solution:-

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$-3x_1 - 25x_2 + 4x_3 = 0$$
$$6x_1 + x_2 - 8x_3 = 0.$$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -25 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]$$

First we will convert this matrix to row reduced form.

* $R_2 = R_2 + R_1.$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 30 & 0 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]$$

* $R_3 = R_3 - 2R_1.$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 30 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right]$$

★ $R_1 = \frac{R_1}{3}$.

$$\left[\begin{array}{ccc|c} 1 & 5/3 & -4/3 & 0 \\ 0 & 30 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right]$$

★ $R_2 = \frac{R_1}{30}$.

$$\left[\begin{array}{ccc|c} 1 & 5/3 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right]$$

★ $R_1 = R_1 - \left(\frac{5}{3}\right) R_2$.

$$\left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right]$$

★ $R_3 = R_3 + 9R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now writing this in equation form.

$$x_1 \quad -\frac{4}{3}x_3 = 0$$
$$x_2 \quad = 0$$
$$\underset{\substack{\downarrow \\ \text{free variable}}}{0x_3} = 0$$

$$x_1 - \frac{4}{3}x_3 = 0$$
$$1x_2 = 0$$
$$x_3 = x_3$$

$$x_1 = \frac{4}{3}x_3$$
$$x_2 = 0$$
$$x_3 = x_3$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3\ x_3 \\ 0 \\ x_3 \end{bmatrix} = X_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$ Answer

Question 6 Reduce the matrix to Normal form and find it's rank.

$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 0 \end{bmatrix}$$

Solution:-

Performing row operations.

★ $R_2 = R_2 - 3R_1$.

$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 1 & 3 & 4 & 0 \end{bmatrix}$$

* $R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

* $R_3 = R_3 - \left(\frac{1}{2}\right) R_2$

$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is reduced to normal form.

The rank of a matrix is the number of non-zero rows in the reduced matrix. So the rank is 2.