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Sec ~ A

Subject ~ Differential Equation

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Question No # 1

Part # 1

$w = \sin(x+ct) + \cos(2x+2ct)$

Solution:~

$$w = \sin(x+ct) + \cos(2x+2ct)$$

$$\frac{\partial w}{\partial t} = \cos(x+ct) + c - \sin(2x+2ct) + 2c$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) + c^2 - \cos(2x+2ct) + 4c^2 \rightarrow 1$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - \sin(2x+2ct) + 2$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$= [-\sin(x+ct) - 4\cos(x+2ct)$$

$$\boxed{c^2 = \partial^2 w / \partial x^2 \text{ Ans}}$$

Question No 1

Part (b)

Ans Given data :-

$w = \tan(2x + ct)$

Required :- To check if it the solution of given eq or not

Solution :- $w = \tan(2x + ct)$

Partial diff w.r.t "x"

Now

$$\frac{\delta w}{\delta t} = (\sec^2(2x + ct)$$

again Diff :-

$$\frac{\delta^2 w}{\delta t^2} = \frac{\partial}{\delta t}((\sec^2(2x + ct))$$

$$= 2c^2 \cdot 2\sec^2(2x + ct)\tan(2x + ct)$$

$$\frac{\delta w}{\delta x} = 2\sec^2(2x+ct)$$

and

$$\frac{\delta^2 w}{\delta x^2} = 4\sec^2(2x+ct)\tan(2x+ct)$$

$$\Rightarrow 4c^2\sec^2(2x+ct)\tan(2x+ct)$$

$$\Rightarrow 4c^2\sec^2(2x+ct)\tan(2x+ct)$$

$$\Rightarrow 0=0$$

Satisfied;

Question # 02

Given function is

$$f(x) = \begin{cases} x; & -\pi < x \le 0 \\ 2x; & 0 \le x \le \pi \end{cases}$$

Solution:

We have find the Fourier Co-efficients, a_0, a_n and b_n

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{\pi^2}{2}\right] + \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right]$$

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$$a_0 = \frac{-\pi}{2} + \pi = \frac{\pi}{2} \quad \text{— ①}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} (x \cos nx \, dx) + \frac{1}{\pi} \int_{0}^{\pi} (2x \cos nx) \, dx$$

$$= \frac{1}{\pi} \left[x \left(\frac{\sin nx}{n} \right) - \left(-\frac{\cos nx}{n^2} \right) \right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi} \left[x \left(\frac{\sin nx}{n} \right) \cdot \left(-\frac{\cos nx}{n^2} \right] \right._{0}^{\pi}$$

$$a_n = \frac{1}{\pi} \left[\frac{\cos(0)}{n^2} - \frac{\cos nx}{n^2} \right] + \frac{2}{\pi} \left[\frac{\cos nx}{n^2} - \frac{\cos(0)}{n^2} \right]$$

$$= \frac{1}{\pi} \left[\frac{1 - (-1)^n + 2(-1)^n - 2}{n^2} \right]$$

$$= \frac{(-1)^n - 1}{\pi n^2}$$

So

$$a_n = \begin{cases} -\frac{2}{\pi n^2} ; & \text{if } n \text{ is odd} \\ 0 ; & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin nx \, dx = \frac{1}{\pi}$$

$$\int_{-\pi}^{0} x \sin n \, dx + \frac{2}{\pi} \int_{0}^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi} \left(x \left(-\frac{\cos nx}{n} \right) - \left(-\frac{\sin nx}{n^2} \right) \right)_0^{\pi}$$

$$b_n = \frac{1}{\pi} \left[-\frac{\pi \cos n\pi}{n} \right] + \frac{2}{\pi} \left[\frac{-\pi \cos n\pi}{n} \right]$$

$$= -\frac{3 \cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}$$

So the required Furrier series is

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos (2x-1)x}{(2n-1)^2} + 3 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{2}$$

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Question No #3

Solve the initial value problem

$y'' - 4y' + 13y = 8\sin 3x, \; y(0) = 1$ and $y'(0) = 2$

Solution:

$y'' - 4y' + 13y = 8\sin 3x, \; y(0) = 1$ and $y'(0) = 2$

Associated Homogenous eq of (1) is

$$y'' - 4y' + 13y = 0 \rightarrow (2)$$

change (2) into Auxiliary equation

Put $y = m$ in (2)

$m^2 - 4m + 13 = 0$

Use Quadratic Formula

$a = 1, \; b = -4, \; c = 13$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm \sqrt{36i}}{2} = \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

$m_1 = 2+3i$
$m_2 = 2-3i$

$y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$

let

$y_p = A \cos 3x + B \sin 3x \longrightarrow *$

Diff w.r.t x:

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$yp' = -3A\sin 3x + 3B\cos 3x$

Again Diff. w.r.t to "x"

Put in (1)

$\rightarrow (-9A\cos 3x - 9B\sin 3x)$

$-4(-3A\sin 3x + 3B\cos 3x)$

$+13(A\cos B + B\sin 3x) + B\sin 3x$

$\Rightarrow -9A\cos 3x - 12B\cos 3x + 13A\cos 3x$

$-9B\sin 3x + 12A\sin 3x + 13B\sin 3x - 8\sin 3x$

$\rightarrow (-9A - 12B + 13A)\cos 3x + (-9B + 12A + 13B)$

$\sin 3x = 8\sin 3x$

$\Rightarrow (4A - 12B)\cos 3x + (4B + 12A)\sin 3x$

$= 18\sin 3x$

Comparing co-efficient

$\sin 3x \Rightarrow 4B + 12A = 8$ (a)

$\cos 3x \Rightarrow 4A - 12B = 0$

$\Rightarrow 4A = 12B$

$A = 3B \rightarrow$ (b)

Put (b) in (a)

$4B + 12(3B) = 8$

$4B + 36B = 8$

$40B = 8$

$B = \frac{1}{5} \rightarrow$ (c)

Put c and b $a = \frac{3}{5} \rightarrow$ (d)

Put (c) and (d) in (*)

$y_p = \frac{3}{5} \cos 3x + \frac{1}{5} \sin 3x \rightarrow$ (B)

The G. Solution is

$y = y_c + y_p$

$$y = e^{2x}\left(c_1\cos 3x + c_2\sin 3x\right) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \rightarrow \text{(C)}$$

Now we need to find the values of c_1 and c_2 for this

Put $x=0$ and $y=1$ in (C)

$$1 = e^{x(0)}\left(c_1\cos 3(0) + c_2\sin 3(0)\right) + \frac{3}{5}\cos 3(0) + \frac{1}{5}\sin (3)(0)$$

$$1 = c_1(1) + c_2(0) + \frac{3}{5}(1) + \frac{1}{5}(0)$$

$$1 = c_1 + \frac{3}{5}$$

$$c_1 = 1 - \frac{3}{5}$$

$$c_1 = \frac{2}{5} \rightarrow **$$

Now Diff "C" with r "x"

$$y' = c_1\left(2e^{2x}\cos 3x - 3e^{3x}\sin 3x\right) + c_2\left(2e^{3x}\sin 3x + 3e^{2x}\cos 3x\right) - \frac{6}{5}\sin 3x + \frac{3}{5}\cos x - \text{(D)}$$

Put $y' = 2,\ x = 0$ in (D)

$$y' = c_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x)$$
$$+ c_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x)$$
$$- \frac{6}{5}\sin 3x + \frac{3}{5}\cos 3x.$$

Put $y' = 2,\ x = 0$

$$2 = c_1(2e^{2(0)}\cos 3(0) - 3e^{2(0)}\sin 3(0))$$
$$- \frac{6}{5}\sin 3(0) + \frac{3}{5}\cos 3(0)$$

$$2 = 1(2) + c_2(3) - 0 + \frac{3}{5}$$

$$2 = 2c_1 + 3c_2 + \frac{3}{5}$$

Put $c_1 = 2/5$

$$2 = \frac{4}{5} + 3/2 + \frac{3}{5}$$

$$2 = \frac{7}{5} + 3c_2$$

$$3c_2 = 2 - \frac{7}{5}$$

$$3c_2 = \frac{3}{5}$$

$$C_2 = \frac{3}{15} \rightarrow ★★★$$

<u>Put ★★ and ★★★ in C</u>

$$y = e^{2x}\left(\frac{2}{5}\cos 3x + \frac{3}{15}\sin 3x\right) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x$$

$$y = \frac{2}{5}e^{2x}\cos 3x + \frac{3}{15}e^{2x}\sin 3x + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x$$

<u>Req Ans</u>

Question No #4

Solution:

It is already in Symbolic form

$$(D^2 - DD')z = \cos x \cos^2 y \rightarrow (a)$$

Put A.E $D^2 - DD' = 0$

As we know

$$\frac{D}{D'} = m \text{ i.e } D = m, D' = 1$$

$$\Rightarrow m^2 - m = 0$$

$$m = 0, 1$$

Therefore

$$C.F = f_1(y) + f_2(y+x)$$

From eq (a)

$$P.I = \frac{1}{D^2 - DD'} \cos x \cos^2 y$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 - DD'} 2 \cos x \cos^2 y$$

As

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$C.F = f_1(y-x) + x f_2(y-x)$$

$$P.I = \frac{1}{D^2+2DD'+D'^2}\left[2(y-x)+\sin(x-y)\right]$$

$$= \frac{1}{(D+D')^2}\left[2(y-x)+\sin(x-y)\right]$$

<u>By General Method</u>

$$m_2 = -1 \; ; \; y-x = c$$

$$= \frac{1}{D+D'}\left[\int [2c+\sin(-c)]\, dx\right]$$

$$= \frac{1}{D+D'}\left[2cx - (\sin c)\, x\right]$$

<u>Replacing by $y-x$</u>

$$= \frac{1}{D+D'}\left[2x(y-x) - x\sin(y-x)\right]$$

<u>Again Put $y-x=c$</u>

$$= \int (2xc - x\sin x)\, dx$$

$$\Rightarrow cx^2 - \frac{x^2}{2}\sin c$$

Replacing by $y-x$

$$= x^2(y-x) - \frac{x^2}{2}\sin(y-x)$$

$$= x^2y - x^3 + \frac{x^2}{2}\sin(x-y)$$

Hence the required solution is

$$z = C.F + P.I = \phi_1(y-x) + x\phi$$

$$= (y-x) + x^2y - x^3 + \frac{1}{2}x^2 \sin(x-y)$$