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Section- A

Subject:- Differential Equations

Assigment # 02

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c) Use Caushy" Euler equation to solve the following problems. (2)

(a) $x^3y''' + 2x^2y' + 2y - 10x + \frac{10}{x}$

Soln Given $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 10x - \frac{10}{x}$

$\therefore \frac{dy}{dx} = D$

$(x^3D^3 + 2x^2D^2 + 2D + 2)y = 10x - 10x^{-1} \longrightarrow (1)$

~~xD~~ Let $x = e^t \Rightarrow t = \ln x$

$xD = \Delta$

$x^2D^2 = \Delta^2 - \Delta$

$x^3D^3 = \Delta(\Delta-1)(\Delta-2)$

Substitute into eq (1)

$(\Delta^3 - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta) + 2)\,y = 10x - 10x^{-1}$

$(\Delta^3 - \Delta^2 + 2)y = 10x + 10x^{-1}$

$\Rightarrow (m^3 - m^2 + 2)y = 10e^t + \frac{10}{e^t}$

Using synthetic devision

$$\begin{array}{r|rrrr} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

$$D^2 - 2D + 2 = 0$$

Quadratic formula for factorization

$a = 1,\ b = -2,\ c = 2$

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= \frac{2(1 \pm i)}{2}$$

$$D = 1 \pm i$$

Since roots are complex.

$$y_c = e^{-x}(c_1\cos t + c_2\sin t).$$

Now particular integration

$$y_p = \frac{1}{D^3+D^2+2} \cdot 10e^t + \frac{1}{D^3-D^2+2} \cdot \frac{10}{e^t}.$$

$$= \frac{10e^t}{1^3-1^2+2} + \frac{10e^{-t}}{1^3-1+2}$$

$$= \frac{10e^t}{2} + \frac{10e^{-t}}{2}$$

$$y_p = 5e^t + 5e^{-t}$$

General sol;

$$y = y_c + y_p$$

$$y = e^{-x}(c_1\cot + c_2\sin t) + 5e^t + 5e^{-t}$$

put $e^t = x$ and $t = \ln x$.

$$y = e^{-x}(c_1\ln x + c_2\sin \ln x) + 5e^x + 5e^{-x}$$

(b) $x^3\dfrac{d^3y}{dx^3} + 4x^2\dfrac{d^2y}{dx^2} - 5x\dfrac{dy}{dx} - 15 = x^4$ 8

Sol:- Let $D = \dfrac{d}{dx}$

$$x^3D^3y + 4x^2D^2y - 5xDy - 15y = x^4$$

$$(x^3D^3 + 4x^2D^2 - 5xD - 15)\,y = x^4 \to (1)$$

Let $x = e^t \Rightarrow t = \ln x$

$$xD = \Delta$$

$$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$$

$$x^3D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$$

Substitute in eq(1).

$$(\Delta^3 - 3\Delta^2 + 2\Delta + 4(\Delta^2 - \Delta) - 5(\Delta) - 15)\,y = e^{4t}$$

$$(\Delta^3 + \Delta^2 - 7\Delta - 15)\,y = e^{4t}$$

using Synthetic division.

$$\begin{array}{c|cccc} & 1 & +1 & -7 & -15 \\ 5 & & 3 & 12 & 15 \\ \hline & 1 & 4 & 5 & \boxed{0} \end{array}$$

$$\Delta^2 + 4\Delta + 5 = 0$$

$\longrightarrow$

using quadratic formula

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$D = 2\frac{(-4 \pm 2i)}{2}$$

$$D = -2 + i$$

$$\Rightarrow y_c = e^{4t}(c_1 \cos t + c_2 \sin t)$$

and $$y_p = \frac{1}{D^3 + D^2 - 7D - 15} e^{4t}$$

$$= \frac{1}{4^3 + 4^2 - 7(4) - 15} e^{4t} \qquad \therefore D = 4$$

$$= \frac{1}{37} e^{4t}$$

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Sol:

$$y = y_c + y_p$$

$$y = (c_1 \cos t + c_2 \sin t) + \frac{1}{37} e^{4t}$$

put. $t = \ln x$ and $x = e^{4t}$

$$y = e^{3x}(c_1 \cos \ln x + c_2 \sin \ln x) + \frac{1}{37} e^{4x}$$

(c) $x^2y'' + 2xy' - 6y = 10x^2$

Sol: Given that

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 6y = 10x^2$$

and

$$y(1) = 1 \text{ and } y'(1) = -6$$

~~put~~ put $\frac{d}{dx} = D$

$$x^2D^2y + 2xDy - 6y = 10x^2$$

$$(x^2D^2 + 2xD - 6)y = 10x^2 \quad \text{--- (1)}$$

Let $xD = \Delta$

$$x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$$

and $x = e^t$ and $\ln x = t$.

So; eq(1) $\Rightarrow$ $(\Delta^2 + \Delta - 6)y = 10e^{2t}$.

Now characteristic equation;

$$D^2+D-6=0$$

$$\Rightarrow D^2+3D-2D-6=0$$

$$\Rightarrow D(D+3)-2(D+3)=0$$

$$\Rightarrow D+3=0,\quad D-2=0$$

$$D=-3,\ D=2$$

Thus roots are real

$y_c = ?$

$$y_c = C_1 e^{-2t} + C_2 e^{2t}$$

$y_p = ?$

$$y_p = \frac{1}{D^2-D-6} \cdot 10^{2t}$$

$$\Rightarrow y_p = \frac{10}{6} e^{2t}$$

Now

$$10 \frac{1}{d/dD\,(D^2-D-6)} e^{2t}$$

$$\Rightarrow 10 \frac{1}{2D+1} e^{2t}$$

$$\Rightarrow y_p = 2te^{2t}$$

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(18)

General sol:-

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{-3t} + c_2 e^{2t} + 2te^{2t}$$

$$\cancel{y = c_1 x^{-3t} + c_2 x^{2t}}$$

$$y = c_1 x^{-3} + c_2 x^2 \ln x \, x^2 \longrightarrow (B)$$

put $y(1) = 1$

$$1 = 1(1)^{-3} + 2(1)^2 + 2\log 1$$

$$1 = c_1 + c_2 \longrightarrow (A)$$

Diffi eq (B) w.r.t x

$$y' = -3x^{-4} + 2c_2 x + \frac{2}{x} x^2 + 4x \log x$$

put $y'(1) = -6$

$$-6 = -3c_1 + 2c_2 + 2 + 0$$

$$-6 = -3c_1 + 2c_2 + 2$$

$$\cancel{-6 - 2}$$

$$-8 = -3c_1 + 2c_2 \longrightarrow (P)$$

multiply eq (A) with 2 and substract from eq (P)

$\longrightarrow$

$2c_1+2c_2=2$
$3c_1+2c_2=-8$

$5c_1=10$

$\boxed{c_1=2.}$

put c_1 in eq (P)

$-8=-3(2)+2c_2$

$-2=2c_2$

$\Rightarrow \boxed{c_2=-1}$

put Values of c_1 and c_2 in eq(B)

$y=2x^{-3}-x^2+2\ln x\,(x^2)$

$y=\frac{2}{x^3}-x^2+2x^2\log x.$

Q(d) $x^2y''+7xy'+5y=x^5$

$y(0)=2$ and $y'(1)=2$

Sol:- Given:- $x^2\frac{d^2y}{dx^2}+7x\frac{dy}{dx}+5y=x^5$

and $y(0)=2,\ y'(1)=2.$

Let $\frac{d}{dx}=D$

$x^2D^2y+7xDy+5y=x^5$

$(x^2D^2+7xD+5)y=x^5 \rightarrow ①$

Let $xD=\Delta$

$x^2D^2=\Delta(\Delta-1)=\Delta^2-\Delta$

and $x=e^t \Rightarrow \ln x=t$ put in eq ①

$(\Delta^2-\Delta+7\Delta+5)y=e^{5t}$

$(\Delta^2+6\Delta+5)y=e^{5t}.$

$\Rightarrow (m^2+6m+5)y=e^{5t}$

Solving By Quadratic formulas

$$\Delta=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\frac{-6\pm\sqrt{6^2-4(1)(5)}}{2(1)} \rightarrow$$

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$$= \frac{-6 \pm \sqrt{36-20}}{2}$$

$$= -6 \pm \sqrt{16}$$

$$= \frac{-6 \pm 4}{2}$$

$$= \frac{2(-3 \pm 2)}{2}$$

$$D = -3 \pm 2 \quad (\text{real})$$

$$y_c = C_1 e^{-5t} + C_2 e^{5t}$$

$$y_p = \frac{1}{D^2 + 6D + 5} e^{5t}$$

$$= \frac{1}{5^2 + 6(5) + 5} e^{5t}$$

$$y_p = \frac{1}{60} e^{5t}$$

Now General solution

$$y = y_c + y_p$$

$$y = c_1 e^{-5t} + c_2 e^{-t} + \frac{1}{60} e^{5t}$$

$$y = c_1 x^{-5} + c_2 x^{-1} + \frac{1}{60} x^5 \longrightarrow (2)$$

put $x = 0$ in eq(2)

and $y(0) = 2$

$$2 = 1(2)^{-5} + 2(2)^{-1} + \frac{1}{60} 2^5$$

$$2 = -32c_1 - 2c_2 + \frac{2}{15}$$

$$\frac{22}{15} = -32c_1 + 2c_2 \rightarrow (a)$$

Diffi eq(2) w.r.t x

$$y' = -5c_1 x^{-6} + c_2 x^{-2} + \frac{1}{12} x^4$$

put $y'(0) = 2$ in above eq

$$2 = 320c_1 + 4c_2 + \frac{4}{3}$$

$$\Rightarrow \frac{2}{3} = 320c_1 + 4c_2 \rightarrow (b)$$

multipy eq(a) by 2 and Subtract from eq (b)

$$-\frac{44}{15} = 64c_1 + 4c_2$$

$$-\frac{44}{15} = 6c_1 + 4c_1$$

$$\frac{2}{3} = 320c_1 + 4c_2$$

$$\frac{34}{15} = -256c_2$$

$$\boxed{c_1 = 580}$$

Put value of C_1 and C_2 in eq(6)

$$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x^5$$

$$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5$$ Ans

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(e) $(x+1)^2y'' - 3(x+1)y' + 4y = x^2$

Sol:- $(x+1)^2\frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2$

Let $\frac{d}{dx} = D$

$(x+1)^2D^2y + 3(x+1)Dy + 4y = x^2$

$((x+1)^2D^2 + 3(x+1)D + 4)y = x^2$

Let $(x+1)D = \Delta$

and $x = e^t$

$(\Delta^2 - \Delta - 3\Delta + 4)y = e^{2t}$

$(\Delta^2 - 4\Delta + 4)y = e^{2t}$

for y_c we find roots

$\Delta^2 - 4\Delta + 4 = 0$

$\Delta^2 - 2\Delta - 2\Delta + 4 = 0$

$\longrightarrow$

Put $x=0$ in eq(2)

and $y(0)=2$

$$2 = 1(2)^{-5} + 2(2)^{-1} + \frac{1}{60}2^{5}$$

$$2 = -32c_1 - 2c_2 + \frac{2}{15}$$

$$\frac{22}{15} = -32c_1 + 2c_2 \rightarrow (a)$$

Diff. eq(2) w.r.t x

$$y' = -5 \cdot x^{-6} + c_2 x^{-2} + \frac{1}{12}x^{4}$$

put $y'(0)=2$ in above eq

$$2 = 320c_1 + 4c_2 + \frac{4}{3}$$

$$\Rightarrow \frac{2}{3} = 320c_1 + 4c_2 \rightarrow (b)$$

multipy eq(a) by 2 and Subtract from eq (b)

$$-\frac{44}{15} = 64c_1 + 4c_2$$

$$\frac{-44}{15} = 6c_1 + 4c_1$$

$$\frac{2}{3} = 320c_1 + 4c_2$$

$$\frac{34}{15} = -256c_2$$

$$\boxed{c_1 = 580}$$

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$D(D-2) - 2(D-2) = 0$

$\Rightarrow D-2=0 \;,\; D-2=0$

$D=2 \;,\; D=2$ real roots

general sol; is :

$y = (c_1 + c_2 x)^{2x}$

$y = (c_1 + c_2 x)^{2x}$

for $y_p = ?$

$$y_p = \frac{1}{D^2 - 4D + 4}$$

$$y_p = \frac{1}{2D-4} e^{2t}$$

put $i = D$

$y_p = \frac{2}{0}$ fail

Diffi again

$y_p = \frac{1}{2} e^{2t}$

$y = (c_1 + c_2 \cdots)^{2t} + e^{2t}$