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Assigment: Differential Equations.

Q No 1:- $x^3y''' + 2x^2y' + 2y = 10x + \frac{10}{x}$.

Solution:-

Put $x = e^t$ then.

$\frac{dx}{dt} = e^t \Rightarrow \frac{dt}{dx} = e^{-t}$.

Now.

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot e^{-t}$.

Or.

$y' = \frac{dy}{dx} = e^{-t} Dy$ $\quad\because \frac{d}{dt} \to D$.

Similarly:

$y'' = e^{-2t}[D(D-1)]y$.

$y''' = e^{-3t}[D(D-1)(D-2)]y$

Using these values in ①.

$e^{3t} \cdot e^{-3t}[D(D-1)(D-2)]y + 2e^{2t} \cdot e^{-2t}[D(D-1)]y + 2y =$
$10e^t + 10e^{-t}$.

$\Rightarrow (D^3 - 3D^2 + 2D)y(2D^2 - 2D)y + 2y = 10e^t + 10e^{-t}$

$\Rightarrow D^3y - D^2y + 2y = 10e^t + 10e^{-t}$

OR.

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 2y = 10e^t + 10e^{-t} \quad \text{(2)}$$

The associated homogenous equation of (2) in.

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 2y = 0.$$

$$\frac{d}{dt^2} = K^2, \quad \frac{d^3}{dy^3} = K^3$$

$$\Rightarrow (K^3y - K^2y + 2y) = 0.$$

$$\Rightarrow (K^3 - K^2 + 2)y = 0.$$

fornon-trivial solution $y \neq 0$.

$$K^3 - K^2 + 2 = 0.$$

$\Rightarrow$ Roots are

$$K = -1, \ 1 \pm 2.$$

$$\Rightarrow y_c(t) = Ae^{-t} + (B\cos t + C\sin t)e^t$$

which is complementry solution.

Q3:- $x^2y'' + 2xy' - 6y = 10x^2$; $y(1) = 1$
& $y'(1) = -6$.

Solution:-

$$x^2y'' + 2xy' - 6y = 10x^2 \longrightarrow (i)$$
$$y'(1) = -6$$
$$y(1) = 1.$$

Let $x = e^t$ i.e $t = \log x$.

Now $xy' = \Delta y \Rightarrow x^2y'' = \Delta(\Delta - 1)y$.

where $\Delta = \frac{d}{dt}$

Then eq① $= (\Delta(\Delta-1) + 2\Delta - 6]y = 10e^{2t}$

$$[\Delta^2 - \Delta + 2\Delta - 6]y = 10e^{2t}.$$

$$(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}$$

$$(\Delta^2 + \Delta - 6)y = 10e^{2t}$$

Char form eq . $\Delta^2 + \Delta - 6 = 0$.

$$\Delta + 3\Delta - 2\Delta - 6 = 0.$$

$$\Delta = -3 + \Delta = 2.$$

Complimentry function.

$$C.F = C_1e^{-3t} + C_2e^{2t}$$

Also P. Integral.

$$P.I = \frac{1}{\Delta^2 + \Delta - 6} 10e^{2t}.$$

$$= 10\frac{1}{(2)^2 + 2 - 6} e^{2t} \text{ replace } \Delta \text{ by } 2.$$

eq (A) $= 1 = c_1 - 1.$

$$\boxed{c_1 = 2}$$

Then (A).

$$\Rightarrow \boxed{y = 3x^{-3} - x^2 + 2x^2 \log x.}$$

Q4:- $x^2y'' + 7xy' + 5y = x^5$;

$y(0) = 2$ & $y'(1) = 2$.

Solution:-

$$x^2y'' + 7xy' + 5y = 5x.$$

$y(0) = 2$

$y'(1) = 2$

Let.

$$x = e^t \Rightarrow t = \log x \ , \ \Delta = \frac{d}{de}$$

Now

$$xy' = \Delta y \Rightarrow x^2y = \Delta(\Delta-1)y.$$

Then.

$$(\Delta(\Delta-1) + 7\Delta + 5)y = e^{5t}.$$

$$(\Delta^2 + 6\Delta + 5)y = e^{5t}$$

$$(\Delta^2 + 6\Delta 5)y = e^{5t}.$$

Char equation. is $(\Delta^2 + 6\Delta 5) = 0$.

$$\Delta^2 + 5\Delta + \Delta + 5 = 0.$$

$$\Delta = -5 \ -1$$

Compure equation.

$$C.P = C_1 e^{-5t} + C_2 e^{-t}$$

eq ② P.I.

$$P.I = \frac{1}{\Delta^2 + 6\Delta + 5} e^{5t}.$$

$$= \frac{1}{5^2+6(5)+5} e^{5t}. \quad \text{replace } \Delta \text{ by } 5.$$

$$= \frac{1}{60} e^{5t}.$$

Then. general solution.

$$y = c_1 e^{-5t} + c_2 e^{-t} + \frac{1}{60} e^{5t}.$$

$$y = c_1 x^{-5} + c_2 x^{-1} + \frac{1}{60} x^5.$$

$$y = -5c_1 x^{-6} - c_2 x^{-2} + \frac{1}{12} x^4$$

$y(0) = 2 \quad x = 0,\ y = 2.$

$$2 = c_1 + c_2 + \frac{1}{60}.$$

$$c_1 + c_2 = \frac{119}{60} \longrightarrow (A).$$

$y'(1) = 2 \quad x = 1,\ y' = 2.$

$$2 = -5c_1 - c_2 + \frac{1}{12}$$

$$-5c_1 - c_2 = \frac{23}{12} \longrightarrow (B).$$

$A + B.$

$$-4c_1 = \frac{234}{60} \Rightarrow c_1 = -\frac{117}{120}$$

Now

$$y = \frac{-117}{120} x^{-5} + c_2 x^{-1} + \frac{1}{60} x^5$$

$C_1 = \frac{-117}{120}$ put in (A).

$$\frac{-117}{120} + C_2 = \frac{119}{60}$$

$$C_2 = \frac{119}{60} + \frac{117}{120}.$$

$$\frac{238 + 117}{120} = \frac{355}{120}.$$

Q5:- $(x+1)^2 y'' - 3(x+1) y' + 4y = x^2$

Solution:-

$$(x+1)^2 y'' - 3(x+1) y' + 4y = x^2 \quad \text{—— (1)}$$

Let.

$$x+1 = e^t \Rightarrow x = e^t - 1.$$

Diff $\log(x+1) = t$.

Also. $(x+1) y' = \Delta y$. $\qquad \frac{d}{dt} = \Delta$.

$(x+1)^2 y'' = \Delta(\Delta-1) y$. $\qquad D = \frac{d}{dx}$.

Then eq(1) $\Rightarrow (\Delta(\Delta-1) - 3\Delta + 4) y = (e^t - 1)^2$.

$$(\Delta^2 - 4\Delta + 4) y = e^{2t} - 2e^t + 1).$$

Char equation is $\Delta^2 - 4\Delta + 4 = 0$.

$$(\Delta - 2)^2 = 0.$$

$$\Delta = 2, 2.$$

Thus complementary function is

$$C.F = (c_1 + c_2 t) e^{2t}.$$

Also particular intergral is.

$$P.I = \frac{1}{(\Delta-2)^2} (e^{2t} - 2e^t + 1).$$

$$= \frac{1}{(\Delta-2)^2} e^{2t} - 2 \frac{1}{(\Delta-2)^2} e^t + \frac{1}{(\Delta-2)^2} \rightarrow (2).$$

Now.

$$\frac{1}{(\Delta-2)^2} e^{2t} = \frac{1}{(2-2)^2} e^{2t} = \frac{1}{0} e^{2t}$$

Case of failure

$$\frac{1}{(\Delta-2)^2} e^{2t} = \frac{t}{2} \frac{1}{(1-2)^2} e^{t} = \frac{t^2}{2} e^{t}.$$

and

$$2\frac{1}{(\Delta-2)^2} e^{t} = 2\frac{1}{(1-2)^2} e^{t} = \cancel{\frac{t^2}{2}e^{t}} \cdot 2e^{t}.$$

and.

$$\frac{1}{(\Delta-2)^2}(1) = \frac{1}{(\Delta-2)^2} e^{0t} = \frac{1}{4}$$

$$eq\ (2) \rightarrow \quad P.I = \frac{1}{2} t^2 e^{2t} - 2e^{t} + \frac{1}{4}$$

Hence complete solution is

$$y = C.F + P.I.$$

$$y = (c_1 + c_2 t)e^{2t} + \frac{1}{2} t^2 e^{2t} - 2e^{t} + \frac{1}{4}$$

Replace value of et.

$$y = (c_1 + c_2 \log(x+1))(x+1)^2 + \frac{1}{2}\left[(\log x + x)^2 (x+1)^2\right] - 2(x+1) + \frac{1}{4}$$

OR

$$c_1 + c_2 \log(x+1)(x+1)^2$$
$$\frac{1}{2}(\log x + x)^2 (x+1)^2] - 2(x+1) + \frac{1}{4}$$

which is required.