Department of Electrical Engineering
Final Exam Assignment
Date: 27/06/2020

Course Details

Course Title: Digital Signal Processing **Module:** 6th
Instructor: **Total Marks:** 50

Student Details

Name: **ABDUL BASIT** **Student ID: 13684**

Q1.	(a)	Determine the response y(n), n ≥ 0, of the system described by the second order difference equation $$y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$$ To the input $x(n) = (-1)^n u(n)$. And the initial conditions are y (-1) = y (-2) = 0.	**Marks 7** **CLO 2**
	(b)	Determine the impulse response and unit step response of the systems described by the difference equation. $$y(n) - 0.7y(n-1) + 0.1y(n-2) = 2x(n) - x(n-2)$$	**Marks 7** **CLO 2**
Q2.	(a)	Determine the causal signal x(n) having the z-transform $$x(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})^2}$$ (Hint: Take inverse z-transform using partial fraction method)	**Marks 6** **CLO 2**
	(b)	Evaluate the inverse z- transform using the complex inversion integral $$X(z) = \frac{1}{1-az^{-1}} \qquad \lvert z\rvert > \lvert a\rvert$$	**Marks 6** **CLO 2**
Q.3	(a)	A two- pole low pass filter has the system response $$H(z) = \frac{b_o}{(1-pz^{-1})^2}$$ Determine the values of b_o and p such that the frequency response H(ω) satisfies the condition H(0) = 1 and $\lvert H(\frac{\pi}{4})\rvert^2 = \frac{1}{2}$.	**Marks 6** **CLO 3**

	(b)	Design a two-pole bandpass filter that has the center of its passband at ω = π/2, zero in its frequency response characteristics at ω = 0 and ω = π and its magnitude response in $\frac{1}{\sqrt{2}}$ at ω = 4π/9.	**Marks** **6** **CLO** **3**
Q 4	(a)	A finite duration sequence of Length L is given as $x(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & otherwise \end{cases}$ Determine the N- point DFT of this sequence for N ≥ L	**Marks** **6** **CLO** **2**
	(b)	Perform the circular convolution of the following two sequences. Solve the problem step by step $x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$ $x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$	**Marks** **6** **CLO** **2**

Name:- Abdul Basit

ID:- 13684.

Question 1 (a)

Sol

Conside the difference equation

$$y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$$

The homogenous equation of the system is

$$y(n) - 4y(n-1) + 4y(n-2) = 0.$$

The characteristic equation of the system is

$$\lambda - 4\lambda^{-1} + 4\lambda^{-2} = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

Determine the root of the characteristic equation

$\lambda^{-2} - 2\lambda - 2\lambda + 4 = 0$

$\lambda(\lambda - 2) - 2(\lambda - 2) = 0$

$(\lambda - 2)(\lambda - 2)$

$\lambda = 2, 2,$ ' Hence.

$y_h(n) = C_1 2^n + C_2 n 2^n$

The particular solution is

$y_p(n) = K(-1)^n u(n)$

By substituting this solution into difference equation we obtain

$K(-1)^n u(n) - 4K(-1)^{n-1} u(n-1) + 4(K(-1)^{n-2} u(x-2) =$
$(-1) u(n) - (-1)^{n-1} u(x-1)$

For $n=2$

$K(1+4+4) = 2$

$K = \frac{2}{9}.$

So $y(n) = \left[c_1 2^n + c_2 n 2^n + \frac{2}{9}(-1)^n\right] u(n)$

From the initial condition we obtain $y(0) = 1$, $y(1) = 2$

then

$$c_1 + \frac{2}{9} = 1$$

$$\Rightarrow c_1 = \frac{7}{9}$$

$$2c_1 + 2c_2 - \frac{2}{9} = 2$$

$$\Rightarrow c_2 = \frac{1}{3}$$

Question 1(b).

Sol

Consider the difference equation

$$y(n) - 0.7y(n-1) + 0.1y(n-2) = 2x(n) - x(n-2)$$

To obtain the homogenous equation

$x(n) = 0$

$$y(n) - 0.7y(n-1) + 0.1y(n-2) = 0$$

Determine the solution to the homogenous $y_h(n) = \lambda^n$

Substitute the solution obtained in the homogenous equation

$$\lambda^n - 0.7\lambda^{n-1} + 0.1\lambda^{n-2} = 0$$

$$\lambda^{n-2}(\lambda^2 - 0.7\lambda + 0.1) = 0$$

$$\lambda^2 - 0.7\lambda + 0.1 = 0$$

Therefore, the roots are.

$(\lambda - 0.5)(\lambda - 0.2) = 0$

There fore, the roots are

$\lambda = \frac{1}{2}, \frac{1}{5}$ hence.

General form of the solution

$y_h(n) = C_1(\lambda_1)^n + C_2(\lambda_2)^n$

$y(n) = C_1(0.2)^n + C_2(0.5)^n$ ---(1)

$\lambda = \frac{1}{2}, \lambda = \frac{1}{5}$ then

$y_h(n) = C_1 \frac{1}{2}^n + C_2 \frac{1}{5}^n$

with $x(n) = \delta(n)$ we have

$y(0) = 2.$

$y(1) - 0.7y(0) = 0$

$y(1) = 1.4$

Hence $C_1 + C_2 = 2$

$\frac{1}{2} C_1 + \frac{1}{5} = 1.4 = \frac{7}{5}$

$$\Rightarrow C_1 + \frac{2}{5} C_2 = \frac{14}{5}$$

These equation yield

$$C_1 = \frac{10}{3}, \quad C_2 = -\frac{4}{3}$$

$$h(n) = \left[\frac{10}{3}\left(\frac{1}{2}\right)^n - \frac{4}{3}\left(\frac{1}{5}\right)^n\right] u[n]$$

Now step response is

$$s(n) = \sum_{K=0}^{n} h(n-K)$$

$$= \frac{10}{3} \sum_{K=0}^{n} \left(\frac{1}{2}\right)^{n-K} - \frac{4}{3} \sum_{K=0}^{n} \left(\frac{1}{5}\right)^{n-K}$$

$$= \frac{10}{3}\left(\frac{1}{2}\right)^n \sum_{K=0}^{n} 2^K - \frac{4}{3}\left(\frac{1}{5}\right)^n \sum_{K=0}^{n} 5^K$$

$$s(n) = \frac{10}{3}\left(\frac{1}{2}\right)^n (2^{n+1} - 1) u(n) - \frac{1}{3}\left(\frac{1}{3}\right)^n (5^{n+1} - 1)$$

$$u(n)$$

x——x——x

Q2(a) Determine the causal signal $x(n)$ having the Z-transform.

$$X(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})^2}$$

(Hint:- Take inverse Z-transform using partial fraction method).

Solution:-

The Z-transform is,

$$X(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})^2}$$

The expression is written as.

$$X(z) = \frac{1}{\left(1-\frac{2}{z}\right)\left(1-\frac{1}{z}\right)^2}$$

$$= \frac{1}{\left(\frac{z-2}{z}\right)\left(\frac{z-1}{z}\right)^2}$$

$$= \frac{1}{\frac{(z-2)(z-1)^2}{z^3}}$$

$$= \frac{z^3}{(z-2)(z-1)^2} \quad \text{------} \quad (1).$$

$X(z)$ has a simple pole at $p_1 = 2$ and a double $p_2 = p_3 = 1$. In such a case the appropiate partial-fraction expansion is.

$$X(z) = \frac{z^3}{(z-2)(z-1)^2} = \frac{A_1}{z-2} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2}.$$

The problem is to determine the coefficient A_1, A_2 and A_3.

We proceed as in the case of distinct pole To determine A_1 we multiply both side of by $(z-2)$ and evaluate the result $z=-2$

$$(z-2)X(z) = A_1 + \frac{z-2}{z-1} A_2 + \frac{z-2}{(z-1)^2} A_3.$$

which we evaluated at $z=2$

$$A_1 = \left.\frac{(z-2)X(z)}{z}\right|_{z=2}$$

$$A_1 = 4$$

$$A_2 = A_1 + \frac{z-2}{z-1}$$

$$A_2 = -3.$$

$$A_3 = A_1 + \frac{z-2}{z-1} A_2$$

$$= -1.$$

Hence $x(n) = [4(2)^n - 3 - n]u(n)$.

Question no 2(b).

Solution:-

We have

$$x(n) = \frac{1}{2\pi j} \oint \frac{z^{n-1}}{1-az^{-1}} dz = \frac{1}{2\pi j} \oint \frac{z^n dz}{z-a}$$

where C is a circle at radius greater than $|a|$. We shall evaluate this integral using with $f(z) = z^n$. We distinguish two cases.

1 If $n \geq 0$, $f(z)$ has only zeros and hence no poles inside C. The only pole inside C is $z = a$. Hence

$$x(n) = f(z_0) = a^n, \quad n \geq 0.$$

2) If $n<0$, $f(z) = z^n$ has an nth-order pole at $z=0$, which is also inside c. Thus there are contributions from both poles. For $n=-1$ we have.

$$x(-1) = \frac{1}{2\pi j} \oint_c \frac{1}{z(z-a)} dz = \frac{1}{z-a}\Big|_{z=0}$$

$$+ \frac{1}{z}\Big|_{z=a} = 0$$

if $n=-2$, we have

$$x(-2) = \frac{1}{2\pi j} \oint_c \frac{1}{z^2(z-a)} dz = \frac{d}{dz}\left(\frac{1}{z-a}\right)\Big|_{z=0}$$

$$+ \frac{1}{z^2}\Big|_{z=a} = 0$$

By containving in the same way we can show that $x(n)=0$ for $n<0$, Thus

$$x(n) = a^n u(n).$$

Question No 3(a).

Solution

At $w=0$ we have

$$H(0) = \frac{b_0}{(1-P)^2} = 1$$

hence

$$b_0 = (1-P)^2$$

At $w = \pi/4$

$$H\left(\frac{\pi}{4}\right) = \frac{(1-P)^2}{(1-Pe^{-j\pi/4})^2}$$

$$= \frac{(1-P)^2}{(1-p\cos(\pi/4)+jp\sin(\pi/4))^2}$$

$$= \frac{(1-P)^2}{(1-P/\sqrt{2}+jP\sqrt{2})^2}$$

Hence

$$\frac{(1-p)^4}{[(1-p/\sqrt{2})^2+p^2/2]^2} = \frac{1}{2}.$$

or equivalently

$$\sqrt{2}(1-p)^2 = 1+p^2-\sqrt{2}p$$

The value of $p=0.32$ satisfies this equation Consequently, the system function for the desired filter is

$$H(z) = \frac{0.46}{(1-0.32z^{-1})^2}.$$

Question No 3(b):

Sol

Clearly, the filter must have poles at

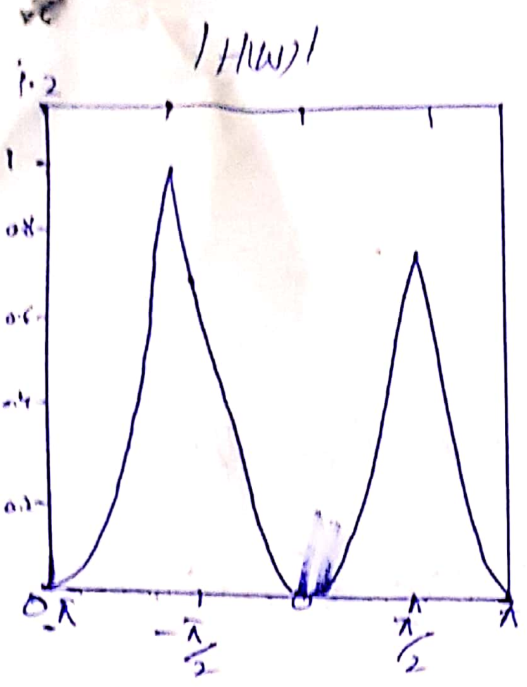
$$p_{1,2} = re^{\pm j\pi/2}.$$

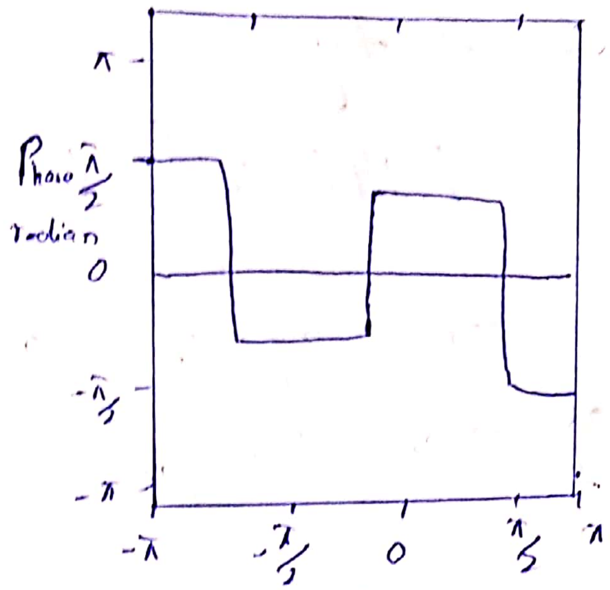
and zeros at $z=1$ and $z=-1$, Consequently, the system function is.

$$H(z) = \frac{G(z-1)(z+1)}{(z-jr)(z+jr)}$$

$$= G\frac{z^2-1}{z^2+r^2}.$$

16 (14)





The gain factor is determined by evaluating the frequency response $H(\omega)$ of the filter at $\omega = \pi/2$ thus we have

$$H\left(\frac{\pi}{2}\right) = G\,\frac{2}{1-r^2} = 1$$

$$G = \frac{1-r^2}{2}.$$

The value of r is determined by evaluating $H(\omega)$ at $\omega = 4\pi/9$. Thus we have.

$$\left|H\left(\frac{4\pi}{9}\right)\right|^2 = \frac{(1-r^2)^2}{4}\;\frac{2-2\cos(8\pi/9)}{1+r^4+2r^2\cos(8\pi/9)} = \frac{1}{2}.$$

or equivalently

$$1.94(1-r^2)^2 = 1 - 1.88r^2 + r^4$$

The value of $r^2 = 0.7$ satisfies this equation. Therefore, the system function for the desired filter is

$$H(z) = \frac{0.15 + z^{-2}}{1+0.7z^{-2}}.$$

Question 4 (a).

Sol

The fourier transform of this sequence is

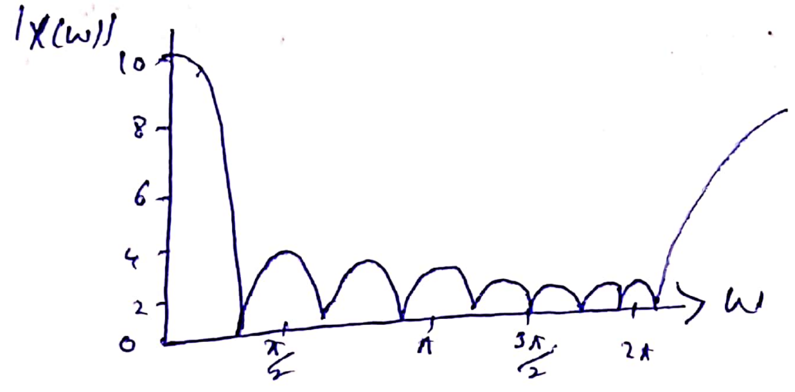
$$X(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n}$$

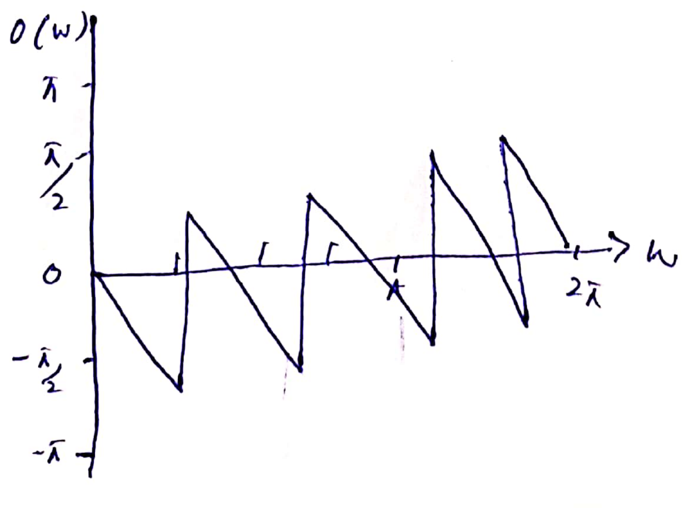
$$= \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

The magnitude and phase of $X(\omega)$ are illustration for $L=10$. The N-point DFT of $x(n)$ is simply $X(\omega)$ evaluated at the set of N equally speed frequencies $\omega_K = 2\pi K/N$, $K = 0, 1, \ldots N-1$ Hence.

$$X(K) = \frac{1 - e^{-j2\pi KL/N}}{1 - e^{-j2\pi K/N}}, \quad K = 0, 1, \ldots, N-1$$

$$= \frac{\sin(\pi KL/N)}{\sin(\pi K/N)} e^{-j\pi K(L-1)/N}.$$





If N is selected such that N = L, then the DFT become

$$X(k) = \begin{cases} L, & k = 0 \\ 0, & k = 1, 2, \ldots, L-1 \end{cases}$$

Thus there is only one non zero value in DFT. This is apparent from observation of $X(\omega)$ since $X(\omega) = 0$ at the frequencies $\omega_k = 2\pi k/L$ $k \neq 0$. The reader should verify that $x(n)$ can be recovered from $X(k)$ by performing an L-point DFT.

Question 4(b).

Solution:-

$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$

$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$.

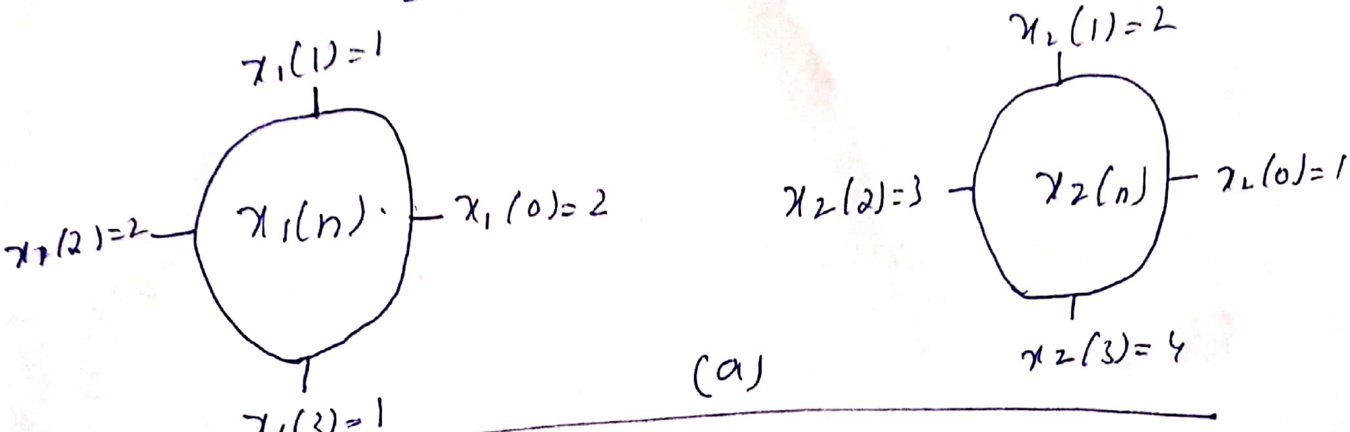
Each sequence consists of four nonzero points. For the purpose of illustrating the operation involved in circular convolution it is desirable to graph each sequence as point on a circle.

Now, $x_3(m)$ is obtained by circularly convolving $x_1(n)$ with $x_2(n)$ as specified by beginning with $m=0$ we have

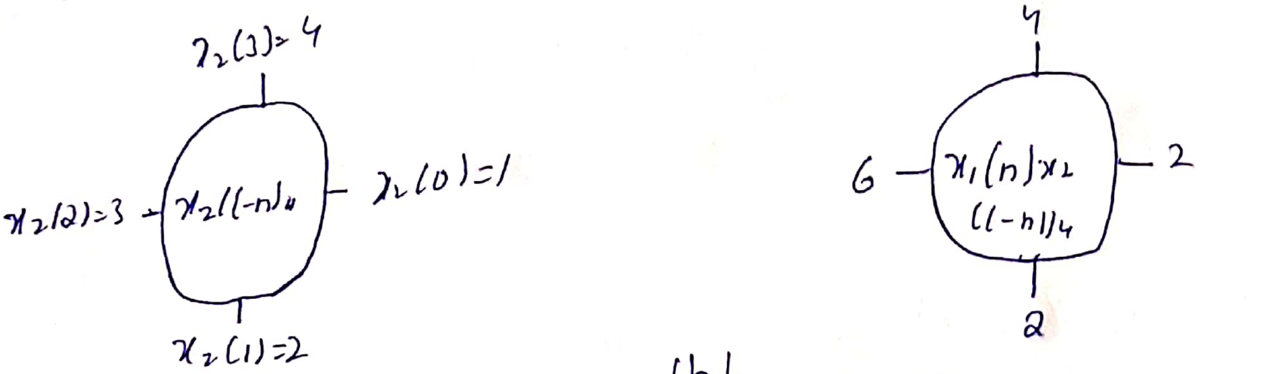
$$x_3(0) = \sum_{n=0}^{3} x_1(n)\, x_2((-n))N.$$

$x((-n))_4$ is simply the sequence $x_2(n)$ folded and graphed on a circle as illustrated. Finally, we sum the values in the product sequence to obtain.

$x_3(0) = 14.$



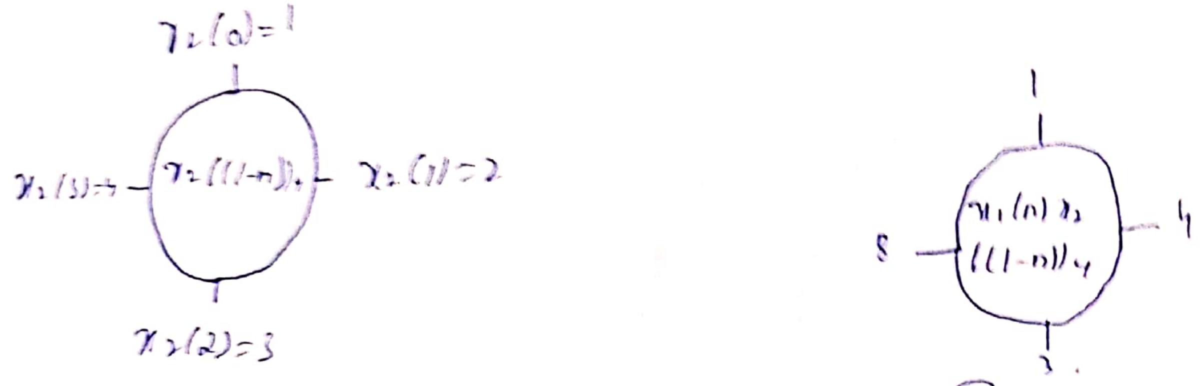
(a)



(b)

Folded sequence

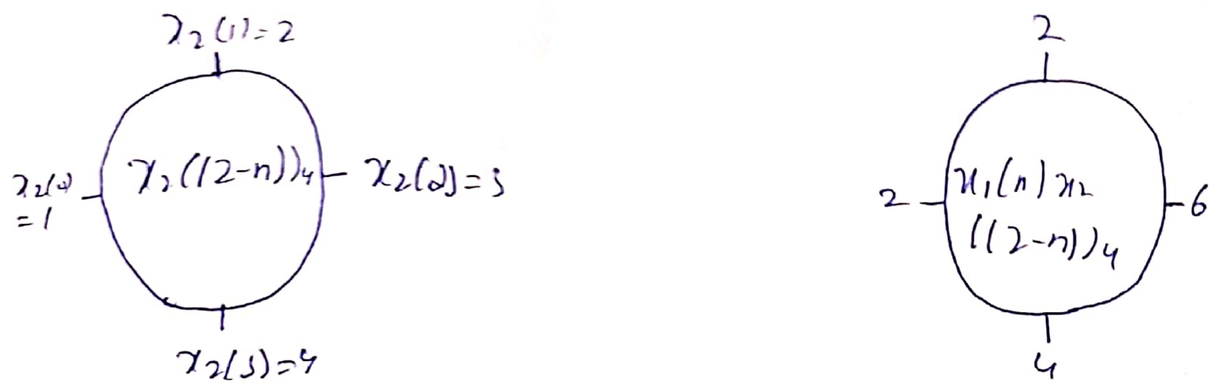
Product sequence



Folded sequence rotated by one Unit in time.

Product Sequence

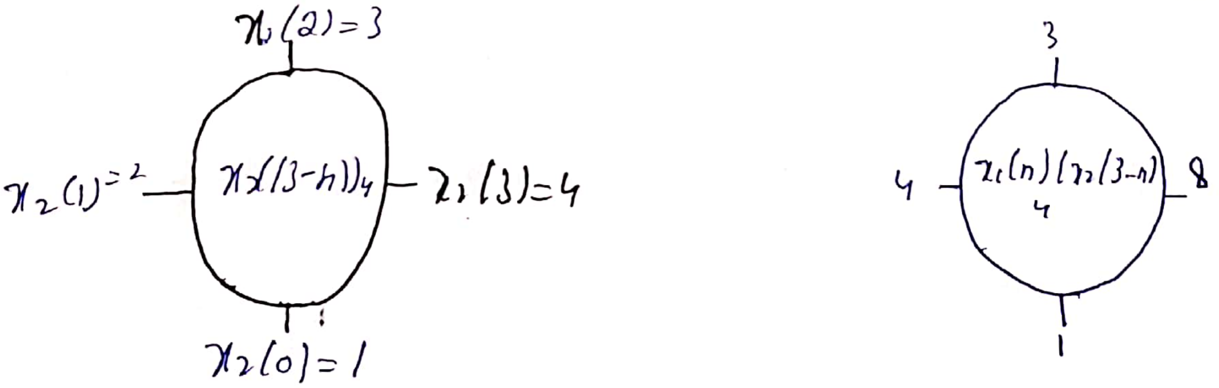
(c)



Folded sequence rotated by unit in time.

(d)

Product sequence



Product sequence

Folded sequence rotated by there unit in time.

(e)

For $m=1$ we have

$$x_3(1) = \sum_{n=0}^{3} x_1(n)\, x_2((1-n))_4.$$

It is easily verified that $x_2((1-n))_4$ is simply the sequence $x_2((-n))_4$ rotated counter clockwise by one unit in time

$x_3(1)=6$

For $m=2$ we have.

$$x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2((2-n))_4$$

Now $x_2((2-n))_4$ is the folded sequence rotated two unit of time in the counterclockwise direction. the product sequence $x_1(n)\, x_2((2-n))_4$.

$x_3(2)=14$

For m=3 we have

$$x_3(3) = \sum_{n=0}^{3} x_1(n)\, x_2((3-n))_4.$$

The folded sequence $x_2((-n))_4$ is now rotated by three unit in time to yield $x_2((3-n))_4$ and the resultant sequence is multiplied by $x_1(n)$ to yield the product sequence

$x_3(3) = 16$.

We observe that if the computation above is continued beyond $m=3$. we simply repeat the sequence of four values obtained above. Therefore, the circular convolution of the two sequence $x_1(n)$ and $x_2(n)$ yield the sequence

$$x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$$