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SECTION: B

SEMESTER: 4th (Spring)

EXAM: Mid

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"DIFFERENTIAL EQUATIONS"

Question No 1

i, The order of matrix A is $m \times p$ and order of B is $p \times n$ Then order of matrix AB is ?

The order of Matrix
$$AB = m \times n.$$

ii, The No. of Non-zero rows in an echelon form?

No. of Non-zero rows = Rank of matrix

iii, If $B = \begin{bmatrix} 1 & 4 \\ 2 & a \end{bmatrix}$ is singular Matrix then $a = ?$

A matrix B, such that $|B| = 0$ is called singular Matrix.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & a \end{bmatrix}$$

$$|B| = \begin{vmatrix} 1 & 4 \\ 2 & a \end{vmatrix} = 0$$

$$(1 \times a) - (4 \times 2) = 0$$

$$a - 8 = 0$$

$$\boxed{a = 8}$$

iv) If $A = \begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$ then $|A| = ?$

$$|A| = \begin{vmatrix} 2i & i \\ i & -i \end{vmatrix}$$

$$= (2i \times (-i)) - (i \times i)$$

$$= -2i^2 - i^2 \qquad i^2 = -1$$

$$= -2(-1) - (-1)$$

$$= 2 + 1$$

$$|A| = 3$$

v) The matrix $A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$ is ?

$A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$ is a scalar matrix.

vi) Solution of $\frac{dy}{dx} + 2xy = y$

$$\frac{dy}{dx} + 2xy = y$$

$$\frac{dy}{dx} = y - 2xy$$

$$\frac{dy}{dx} = y(1 - 2x)$$

$$\frac{1}{y}\,dy = (1-2x)\,dx$$

Integrating

$$\int \frac{1}{y}\,dy = \int (1-2x)\,dx$$

$$\ln y = x - \frac{\cancel{2}x^2}{\cancel{2}} + C$$

$$\ln y = x - x^2 + C$$

$$e^{\ln y} = e^{x-x^2+C}$$

$$\boxed{y = e^{x-x^2+C}}$$

vii, The order and degree of differential equation.

$$\left(\frac{dy}{dx}\right)^3 = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \text{is ?}$$

Order of equation = 1

Degree of equation = 6

viii, The degree and order of $\frac{d^2y}{dx^2} - 4xy = \sin \frac{d^2y}{dx^2}$ is?

order of equation = 2

Degree of equation = undefine

ix, The differential eq $2\frac{dy}{dx}+x^2y=2x+3, y(0)=5$ is?

SOLUTION:

Homogenous differential equation

$$2\frac{dy}{dx}+x^2y=2x+3$$

$$2\frac{dy}{dx}=2x+3-x^2y$$

$$2\frac{dy}{d}=2x+3-x^2y$$

Integrating $2\int 1dy = \int(2x+3-x^2y)dx$

$$2y = \frac{\cancel{2}x^2}{\cancel{2}}+3x-\frac{x^3y}{3}+c$$

$$2y+\frac{x^3y}{3} = x^2+3x+c$$

$$y(2+x^3/3)= x^2+3x+C$$

$$y\left(\frac{6+x^3}{3}\right) = x^2+3x+C$$

$$y = \frac{3(x^2+3x+c)}{6+x^3} \quad \text{— A}$$

Put $y=5$ and $x=0$

$$5 = \frac{\cancel{3}(0+0+c)}{\cancel{6}}$$

$$5 = \frac{1}{2}c$$

$$C = 10$$

Put in equation A.

$$y=\frac{3(x^2+3x+10)}{6+x^2}$$

$$\boxed{y=\frac{3x^2+9x+30}{6+x^2}}$$

X, $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ is ?

SOLUTION:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1-1 & b-a & b^2-a^2 \\ 1-1 & c-a & c^2-a^2 \end{vmatrix} \begin{matrix} \\ R_2-R_1 \\ R_3-R_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Expand by C_1

$$= 1 \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} - 0 + 0$$

$$= \begin{vmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{vmatrix}$$

Taking $b-a$ and $c-a$ from R_1 and R_2 repectively

$$= (b-a)(c-a) \begin{vmatrix} 1 & (b+a) \\ 1 & (c+a) \end{vmatrix}$$

$$= (b-a)(c-a)\{(c+a)-(b+a)\}$$

$$= (b-a)(c-a)[c+\cancel{a}-b-\cancel{a}]$$

$$= (b-a)(c-a)(c-b)$$

RESULT:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

Question No 2

i) Express the determinant $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ as the product of factors which are linear in a, b, c

SOLUTION:

$$B = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$|B| = a\begin{vmatrix} b^2 & c^2 \\ b^3 & c^3 \end{vmatrix} - b\begin{vmatrix} a^2 & c^2 \\ a^3 & c^3 \end{vmatrix} + c\begin{vmatrix} a^2 & b^2 \\ a^3 & b^3 \end{vmatrix}$$

$$|B| = a(b^2c^3 - c^2b^3) - b(a^2c^3 - a^3c^2) + c(a^2b^3 - a^3b^2)$$

$$|B| = ab^2c^3 - ac^2b^3 - a^2bc^3 + a^3bc^2 + a^2cb^3 - a^3cb^2$$

Taking abc as common

$$= abc(bc^2 - cb^2 - ac^2 + a^2c + ab^2 - a^2b)$$

$$= abc[cb(c-b) - ac(c-a) + ab(b-a)]$$

RESULT:

$$|B| = abc[cb(c-b) - ac(c-a) + ab(b-a)]$$

PART - ii

Find the eigen value

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

SOLUTION:

$$A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$= \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & -1 & -1 & 0 \\ -1 & 3-\lambda & -1 & -1 \\ -1 & -1 & 3-\lambda & -1 \\ 0 & -1 & -1 & 2-\lambda \end{bmatrix}$$

expand by R_1

$$+2-\lambda \begin{bmatrix} 3-\lambda & -1 & -1 \\ -1 & 3-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{bmatrix} - (-1) \begin{bmatrix} -1 & -1 & -1 \\ -1 & 3-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix}$$

$$+(-1) \begin{bmatrix} -1 & 3-\lambda & -1 \\ -1 & -1 & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix} + 0 = 0 \text{ — } (A)$$

$$\rightarrow \begin{bmatrix} 3-\lambda & -1 & -1 \\ -1 & 3-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{bmatrix}$$

expand by R_1

$$3-\lambda \begin{vmatrix} 3-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ -1 & 2-\lambda \end{vmatrix} + (-1) \begin{vmatrix} -1 & 3-\lambda \\ -1 & -1 \end{vmatrix} = 0$$

$$3-\lambda[(3-\lambda)(2-\lambda)-(-1)(-1)]+1[(-1)(2-\lambda)-(-1)(-1)]$$
$$-1[(-1)(-1)-(3-\lambda)(-1)]=0$$

$$(3-\lambda)(6-3\lambda-2\lambda+\lambda^2-1)+1(-2+\lambda-1)-1(1+3-\lambda)=0$$

$$(3-\lambda)(5-5\lambda+\lambda^2)+(-3+\lambda)-(4-\lambda)=0$$

$$15-15\lambda+3\lambda^2-5\lambda+5\lambda^2-\lambda^3-3+\lambda-4+\lambda=0$$
$$-\lambda^3+8\lambda^2-18\lambda+8=0 \quad \text{— } 1$$

$$\rightarrow +1\begin{vmatrix} -1 & -1 & -1 \\ -1 & 3-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix}$$ Expand by R_1

$$-1\begin{vmatrix} 3-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix}-(-1)\begin{vmatrix} -1 & -1 \\ 0 & 2-\lambda \end{vmatrix}+(-1)\begin{vmatrix} -1 & 3-\lambda \\ 0 & -1 \end{vmatrix}=0$$

$$-1[(3-\lambda)(2-\lambda)-(-1)(-1)]+1[(-1)(2-\lambda)-(0)]-1[(-1)(-1)]=0$$

$$-1[6-3\lambda-2\lambda+\lambda^2-1]+[-2+\lambda]-1[1]=0$$

$$-6+3\lambda+2\lambda-\lambda^2+1-2+\lambda-1=0$$

$$-\lambda^2+6\lambda-8=0 \quad \text{— } 2$$

$$\rightarrow \; -1\begin{bmatrix} -1 & 3-\lambda & -1 \\ -1 & -1 & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix} \text{ expand by } C_1$$

$$-1\Big[-1\begin{vmatrix} -1 & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 3-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} + 0 = 0$$

$$-1\{-1[(-1)(2-\lambda)-(-1)(-1)] + 1[(3-\lambda)(2-\lambda)-(-1)(-1)]\} = 0$$

$$-1[-1[-2+\lambda-1) + 1(6-3\lambda-2\lambda+\lambda^2-1)] = 0$$

$$-1(3-\lambda+5-5\lambda+\lambda^2) = 0$$

$$-8+6\lambda-\lambda^2 = 0$$

or

$$-\lambda^2+6\lambda-8 = 0 \quad \text{— } 3$$

Putting equation 1, 2, 3 in A

$$(2-\lambda)(-\lambda^3+8\lambda^2-18\lambda+8)-\lambda^2+6\lambda-8-\lambda^2+6\lambda-8=0$$

$$-2\lambda^3+16\lambda^2-36\lambda+16+\lambda^4-8\lambda^3+18\lambda^2-8\lambda-\lambda^2+6\lambda-8-\lambda^2+6\lambda-8=0$$

$$\lambda^4-2\lambda^3-8\lambda^3+16\lambda^2+18\lambda^2-\lambda^2-\lambda^2-36\lambda-8\lambda+6\lambda+6\lambda+16-8-8=0$$

$$\lambda^4-10\lambda^3+32\lambda^2-32\lambda = 0$$

By synthetic division

	1	-10	32	-32
2		2	-16	32
	1	-8	+16	0

$$(\lambda-2)(\lambda^3 - 8\lambda^2 + 16\lambda)$$

$$\lambda(\lambda-2)(\lambda^2-8\lambda+16)=0$$

$\boxed{\lambda=0}$ $\lambda-2=0$ $\lambda^2-8\lambda+16=0$

$\boxed{\lambda=2}$

By factorization

$$\lambda^2-4\lambda-4\lambda+16=0$$

$$\lambda(\lambda-4)-4(\lambda-4)=0$$

$$(\lambda-4)(\lambda-4)=0$$

$\boxed{\lambda=4}$, $\boxed{\lambda=4}$

RESULT:

$\lambda=0$

$\lambda=2$

$\lambda=4$

$\lambda=4$

Question No 3

The rate of change in the form of differential equation is given by $(x^2+3y^2)dx - 2xy\,dy = 0$ Find the general solution at $x = 2$ and $y = 6$.

SOLUTION:

$$(x^2+3y^2)dx - 2xy\,dy = 0$$

$$(x^2+3y^2)\,dx = 2xy\,dy$$

$$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$$

$$\frac{dy}{dx} = \frac{x^{\cancel{2}}}{2\cancel{x}y} + \frac{3y^{\cancel{2}}}{2x\cancel{y}}$$

$$\frac{dy}{dx} = \frac{x}{2y} + \frac{3y}{2x}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right) \longrightarrow \text{A}$$

comparing eq A with $\frac{dy}{dx} = g(y/x)$

eq A is homogenous eq of degree 1

Put $\boxed{\frac{y}{x} = V}$ or $\boxed{y = Vx}$

differentiating w.r.t x

$$\boxed{\frac{dy}{dx} = V + x\frac{dV}{dx}}$$

Put in eq A

$$V + x\frac{dV}{dx} = \frac{1}{2}\left[\frac{1}{V} + 3V\right]$$

$$2V + 2x\frac{dV}{dx} = \frac{1}{V} + 3V$$

$$2x\frac{dV}{dx} = \frac{1}{V} + 3V - 2V$$

$$2x\frac{dV}{dx} = \frac{1}{V} + V$$

$$2x\frac{dV}{dx} = \frac{1+V^2}{V}$$

$$\frac{2V}{1+V^2}\,dV = \frac{1}{x}\,dx$$

Ingetrating

$$\int \frac{2V}{1+V^2}\,dV = \int \frac{1}{x}\,dx$$

$\ln|1+v^2| = \ln x + \ln C$ $\quad \because \ln m + \ln n = \ln mn$

$\cancel{\ln}|1+v^2| = \cancel{\ln} xC$

$\boxed{1+v^2 = xC}$ — B

now reput $v = \frac{y}{x}$

$$1 + \left(\frac{y}{x}\right)^2 = xC$$

$$1 + \frac{y^2}{x^2} = xC$$

$$\frac{x^2+y^2}{x^2} = xC$$

$$\boxed{x^2 + y^2 = x^3 C} \text{ — C}$$

Intial values $x = 2$, $y = 6$

$$2^2 + 6^2 = 2^3 C$$

$$4 + 36 = 8C$$

$$\boxed{C = 5}$$

Put in eq C

$$x^2 + y^2 = x^3 \times 5$$

$$y^2 = 5x^3 - x^2$$

$$y^2 = x^2(5x-1)$$

Taking under-root

$$\sqrt{\cancel{y^2}} = \sqrt{x^2(5x-1)}$$

$$y = \pm x\sqrt{5x-1}$$

RESULT:

$$y = +x\sqrt{5x-1}$$

$$y = -x\sqrt{5x-1}$$

are the required solutions.