Day: M T W T F S (1) Date: / /

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Subject Differential Equations

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Exam Final

Department CS (SE)

Q1/.
(a)

2nd order linear homogenous / Non homogenous differential equation

A differential equation of any order is homogenous if once ~~order~~ ~~is~~ ~~homogenous~~ all the terms involving the unknown function are collected together on one side of the equation, the other side is identically zero.

i.e $y'' - 2y' + y = 0$ is 2nd order of homogenous DE.

A differential equation of any order is non-homogenous if once all the terms involving the unknown functions are collected together on one side, identically zero.

i.e. $y'' - 2y' + y = u.$

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(B) Q) Solve the following 2nd order linear homogeneous/non homogeneous DE?

Q1 A/B (i) $4y'' - 6y' + 7y = 0$

Let's find the root of the characteristics equation:

$4\lambda^2 - 6\lambda + 7 = 0 \Rightarrow \lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(4)(7)}}{2(4)}$

$\lambda = \frac{6 \pm \sqrt{36 - 112}}{8}$

$\lambda = \frac{6 \pm \sqrt{-76}}{8}$

$\lambda = 6/8 + \frac{2\sqrt{19}\, i}{8}$

$\lambda = \frac{3}{4} \pm \frac{\sqrt{19}}{4} i$

$\lambda_1 = 3/4 + \frac{\sqrt{19}\, i}{4}, \lambda_2 = 3/4 - \frac{\sqrt{19}\, i}{4}$

So it has the complex conjugate roots.

$Q_1(x) = e^{\lambda_1 x} \cos^{\lambda_1(x)}$

$Q_2(x) = e^{\lambda_2 x} \sin^{\lambda_2(x)}$

$y = C_1 e^{3/4 x} \cos^{i\sqrt{19}/4 (x)} + e^{3x/4} \sin^{i\sqrt{19}/4 (x)} C_2$

Q1(B) ii $y'' - 4y' - 12y = 3e^{5x}$

Solution The characteristic equation and its roots.

$$y^2 - 4y - 12 = (y-6)(y+2) = 0$$

$$y_1 = -2, \quad y_2 = 6.$$

The complementary solution is then,

$$y_c(t) = c_1 e^{-2t} + c_2 e^{6t}.$$

Q2 Solve the following IVP for the 2nd order linear equations.

(i) $16y'' - 40y' + 25y = 0 \quad y(0) = 3 \quad y'(0) = -9/4$

Solution

The characteristic equation and its root are given below.

$$16y^2 - 40y + 25 = (4y - 5)^2 = 0 \quad y_1 = 5/4, \; y_2 = 5/4$$

The general solution and its derivative are

$$y(t) = c_1 e^{5t/4} + c_2 t e^{5t/4}$$

$$y'(t) = 5/4\, c_1 e^{5t/4} + c_2 e^{5t/4} + 5/4\, c_2 t e^{5t/4}$$

Putting in the initial condition

$$3 = y(0) = c_1$$

$$-9/4 = y'(0) = 5/4\, c_1 + c_2$$

The solution for ivp is then

$$y^t = 3e^{5t/4} - 6te^{5t/4}$$

part ii $y'' + 14y' + 49y = 0 \quad y(-4) = -1 \quad y'(-4) = 5$

Solution

The characteristic equation and its roots are:

$$y^2 + 14y + 49 = (y+7)^2 = 0 \quad y_1 = 7, \; y_2 = -7$$

The general solution and its derivative are

$$y(t) = c_1 e^{-7t} + c_2 t e^{-7t}$$

$$y'^t = -7c_1 e^{-7t} + c_2 e^{-7t} - 7c_2 t e^{-7t}$$

Putting in the initial condition.

$$-1 = y(-4) = c_1 e^{28} - 4c_2 e^{28}$$

$$5 = y'(-4) = -7c_1 e^{28} + c_2 e^{28} + 28 c_2 e^{28}$$

$$= -7c_1 e^{28} + 29 c_2 e^{28}$$

It gives the following constant by solving:-

$$c_1 = -9e^{-28}$$

$$c_2 = -2e^{-28}$$

The solution for IVP is

$$y(t) = -9e^{-28} e^{-7t} - 2te^{-28} e^{-7t}$$

$$y(t) = -9e^{-7(t-4)} - 2te^{-7(t+4)}.$$

(iii) $y'' - 4y' + 9y = 0 \quad y(0) = 0, \; y'(0) = -8$

The characteristic equation for this DE is

$$y^2 - 4y + 9 = 0$$

The roots of the equation are

$$y_1 = 2 \pm \sqrt{5}\, i$$

$$y_2 = 2 \pm \sqrt{5}\, i$$

The general solution to the differential equation is then

$$y(t) = C_1 e^{2t} \cos(\sqrt{5}t) + C_2 e^{2t} \sin(\sqrt{5}t)$$

Applying initial condition along with derivatives

$$y(t) = C_2 e^{2t} \sin(\sqrt{5}t)$$

$$y'(t) = 2C_2 e^{2t} \sin(\sqrt{5}t) + 2\sqrt{5}C_2 e^{2t} \cos(\sqrt{5}t)$$

$$-8 = y'(0) = \sqrt{5}(2) = C_2 = -8/\sqrt{5}$$

Solution is then

$$y(t) = -\frac{8}{\sqrt{5}} e^{2t} \sin$$

Q2 Part iv

$$y'' - 8y' + 17y = 0 \quad y(0) = -4, \; y'(0) = -1$$

The characteristic equation and its roots are:

$$y^2 - 8y + 17 = 0$$

$$y_1 = 4 + i$$

$$y_2 = 4 - i$$

The general solution as well as derivative is:

$$y(t) = C_1 e^{4t} \cos(t) + C_2 e^{4t} \sin(t)$$

$$y'(t) = 4C_1 e^{4t} \cos(t) - C_1 e^{4t} \cos(t) + 4C_2 e^{4t} \sin(t) + C_2 e^{4t} \cos(t)$$

By applying the initial condition gives the following.

$$-y = y(0) = c_1$$
$$-1 = y'(0) = 4c_1 + C_2$$

The solution is then

$$y(t) = -4e^{4t}\cos(t) + 15e^{4t}\sin(t)$$

Q3) Define laplace tranform along with example.? Find the laplace transforms of the given functions.

(i) $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

Q3 part ii $g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

$$g(s) = 4\frac{s}{s^2+(4)^2} - 9\frac{4}{s^2+(4)^2} + 2\frac{s}{s^2+(10)^2}$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

Q3 Part 3 $H(t) = e^{3t} + \cos(6t) - e^{3t}\cos(6t)$

solution

$$H(s) = \frac{1}{s-3} + \frac{s}{s^2+(6)^2} - \frac{s-3}{(s-3)^2+(6)^2}$$

$$= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

Q4) Solve the following IVP using laplace forms.

(i) $y'' - 10y' + 9y = 5t, \quad y(0) = -1, \quad y'(0) = 2.$

Solution

Taking transform of every term

$$\mathcal{L}\{y''\} - 10\,\mathcal{L}\{y'\} + 9\,\mathcal{L}(y) = \mathcal{L}\{5t\}$$

By formulas, we got.

$$s^2Y(s) - sy(0) - y'(0) - 10(sY(s) - y(0)) + 9Y(s) = 5/s^2$$

Putting in the initial conditions

$$(s^2 - 10s + 9)\,Y(s) + s - 12 = 5/s^2$$

Solve for $y(s)$:

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

$$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

The partial fraction of transform will be

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

$$5 + 12s^2 - s^3 = A \cdot s(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$$

Solving for constants:

$s = 0$ $\quad 5 = 9B \quad \Rightarrow B = 5/9$

$s = 1$ $\quad 16 = -8D \quad \Rightarrow D = -2$

$s = 9$ $\quad 248 = 648C \quad \Rightarrow C = 31/81$

$s = 2$ $\quad 45 = -14A + \frac{4345}{81} \quad \Rightarrow A = 50/81$

Plugging in the constant gives:

$$Y(s) = \frac{50/81}{s} + \frac{5/9}{s^2} + \frac{31/81}{s-9} - \frac{2}{s-1}$$

By taking the inverse transform the solution is

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^{t}.$$

Q4 (ii) part (ii)

$$y'' - 6y' + 15y = 2\sin(3t),\ y(0) = -1,\ y'(0) = -4$$

Solution:

Taking the laplace transform of everything and plug in initial conditions.

$(\cancel{s^2 - 6s + 15})$

$$s^2 Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 15Y(s) = 2\frac{3}{s^2+9}$$

$$(s^2 - 6s + 15)Y(s) + s - 2 = \frac{6}{s^2+9}$$

$$Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)}$$

Lets set to partial fraction decomposition

$$Y(s) = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-6s+15}$$

now setting numerators equal gives

$$-s^3 + 2s^2 - 9s + 24 = (As+B)(s^2-6s+15) + (Cs+D)(s^2+9)$$
$$= (A+C)s^3 + (-6A+B+D)s^2 + (15A-6B+9C)s + 15B + 9D$$

Solving for constants:

$$\left.\begin{aligned} s^3 &: A + C = -1 \\ s^2 &: -6A + B + D = 2 \\ s^1 &: 15A - 6B + 9C = -9 \\ s^0 &= 15B + 9D = 27 \end{aligned}\right\} \quad A = \frac{1}{10} \quad B = 1/10 \quad C = \frac{-11}{10} \quad D = 5/2$$

Plugging in the constant gives.

$$Y(s) = 1/10 \left(\frac{s+1}{s^2+9} + \frac{-11s+25}{s^2-6s+15} \right)$$

$$= \frac{1}{10} \left(\frac{s}{s^2+9} + \frac{1\frac{3}{3}}{s^2+9} - \frac{11(s-3}{(s-3)^2+6} - \frac{8\frac{\sqrt{6}}{\sqrt{6}}}{(s-3)^2+6} \right.$$

Finally, take the inverse transform on a solution will be thus

$$y(t) = \frac{1}{10} \left(\cos(3t) + \frac{1}{3} \sin(3t) - 11e^{3t} \cos(\sqrt{6}t) \right)$$

$$- \frac{8}{\sqrt{6}} e^{3t} \sin(\sqrt{6}t) \Big)$$

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Q3 Part - a

Define Laplace transform along with example.

Ans Laplace Tranform :-

Laplace tranform is integral transform that converts of real variable to the function of complex variable i.e

The laplace transform y of a function $f(t)$ for $t > 0$ is defined by the following integral over ∞

$$y\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

general example is

$$f(s) = \int_0^\infty f(t) \cdot e^{-st}\,dt.$$