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Section # A

Module # 7<sup>th</sup>

Subject # Differential Equation

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Q1) Find the Fourier series representation of  
 $f(t) = 1+t, -\pi \leq t \leq \pi$

Solution:-

$$f(t) = 1+t \quad -\pi \leq t \leq \pi$$

there we use the formula

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \quad \text{eq(1)}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+t) dt$$

$$a_0 = \frac{1}{2\pi} \left[ t + \frac{t^2}{2} \right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{2\pi} \left( \pi - (-\pi) + \frac{\pi^2}{2} - \left( -\frac{\pi^2}{2} \right) \right)$$

$$a_0 = \frac{1}{2\pi} \left( 2\pi + \frac{2\pi^2}{2} \right)$$

$$a_0 = \frac{1}{2\pi} (2\pi + \pi^2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1+t) \cos nt dt$$

$$a_n = \frac{1}{\pi} \left( (1+t) \frac{\sin nt}{n} \right)_{-\pi}^{\pi} - \int \frac{\sin n+d}{n} (1+t) dt$$

$$a_n = \frac{1}{\pi} \left( (1+t) \frac{\sin nt}{n} - \frac{\cos nt}{n^2} \right) \Big|_{-\pi}^{\pi}$$

$$a_n = \frac{1}{n+\pi} \left( \cos n\pi - \cos n(-\pi) \right)$$

$$a_n = \frac{-1}{n+\pi} (-1 - (-1))$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1+t) \sin nt \, dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1+t) \sin nt - \int_{-\pi}^{\pi} \left( \sin nt \frac{d}{dt} (1+t) dt \right)$$

$$b_n = \frac{1}{\pi} \left( \frac{(1+t)(-\cos nt)}{n} \right) \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \left( \frac{\cos nt}{n} (1) \right)$$

$$b_n = \frac{1}{\pi} \left( \frac{-1+t}{n} \cos nt \right) \Big|_{-\pi}^{\pi} + \left( \frac{\sin nt}{n\pi} \right) \Big|_{-\pi}^{\pi}$$

$$b_n = \frac{-1}{n\pi} \left( (1+\pi) \cos n\pi \right) - \left( (1-\pi) \cos n(\pi) \right)$$

$$b_n = \frac{-1}{n\pi} \left( \cancel{\cos n\pi} + \pi \cos n\pi - \cancel{\cos n\pi} + \cos n\pi \right)$$

$$b_n = \frac{-1}{n\pi} \left( 2\pi \cos n\pi \right)$$

$$\text{Here } \cos n\pi = \frac{(-1)^{n+1}}{n}$$

$$b_n = 2/n (-1)^{n+1}$$

$$f(x) = \frac{1}{2\pi} (2\pi + \pi^2) + 0 + \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nt \quad \text{Ans}$$

Question 02

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix}$$

Eigen Values

Solution :-

Step = 1

we have

$$(A - \lambda I)x = 0 \quad A = \text{Given Matrix}$$

Step; 2

We have; The characteristics equation is given by

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 3 & 1-\lambda & 4 \\ 0 & 2 & 2-\lambda \end{vmatrix} = 0$$

Step = 3

$$\lambda^3 - (\text{Sum of diagonal element})\lambda^2 + (\text{sum of diagonal minor})\lambda - |A| = 0 \quad \text{--- B}$$

$$\text{Sum of diagonal elements} = 1 + 1 + 2 = 4$$

$$\text{Sum of diagonal minors} = \begin{vmatrix} 4 & -1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= (-6) + (2) + (1)$$

$$= -6 + 2 + 1$$

$$= -3$$

By putting values in eq (B);

$$\lambda^3 - 4\lambda^2 - 3\lambda - 1 = 0 \quad \text{--- (c)}$$

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 4 \\ 0 & 2 & 2 \end{vmatrix} = 1 \begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} - 0 \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} \\ &= 1(2-8) - 0 + 1(6-0) \\ &= -6 + 6 \\ &= 0 \end{aligned}$$

By putting values in (o)

$$\lambda^3 - 4\lambda^2 - 3\lambda - 0 = 0$$

$$\lambda^3 - 4\lambda^2 - 3 = 0$$

$$\lambda(\lambda^2 - 4\lambda - 3) = 0$$

$$\lambda = 0$$

$$\lambda^2 - 4\lambda - 3 = 0$$

Using Quadratic Formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1$$

$$b = -4$$

$$c = -3$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$$

$$= 4 \pm \frac{\sqrt{16+12}}{2} = \frac{4 \pm \sqrt{28}}{2}$$

$$\lambda = \frac{4 + \sqrt{28}}{2}, \quad \lambda = \frac{4 - \sqrt{28}}{2}$$

we have eigen values;

$$\lambda = \left( 0, \frac{4 + \sqrt{28}}{2}, \frac{4 - \sqrt{28}}{2} \right) \text{ Ans.}$$

Q No3 Solve the following system of linear equations.

$$5x + 0 + 4z + 2m = 3$$

$$x - y + 2z + m = 1$$

$$4x + y + 2z + 0 = 1$$

$$x + y + 2 + m = 0$$

Solution:-

$$\left[ \begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array} \right] \xrightarrow{R_4 R_2}$$

$$\left[ \begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 4 & 1 & 2 & 0 & 1 \\ 0 & 2 & -1 & 0 & -1 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1+4/5 & 1 \\ 0 & -1 & +6/5 & +4/5 & 3/5 \\ 0 & 2 & -1 & 0 & -1 \end{array} \right] \xrightarrow{-1/5 \times R_3}$$

$$\left[ \begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -1 & 6/5 & 4/5 & 3/5 \\ 0 & 0 & 7/5 & 8/5 & 1/5 \end{array} \right] \xrightarrow{5 \times R_3 \text{ and } 5 \times R_4}$$

$$\left[ \begin{array}{cccc|c} 5 & 0 & 4 & 2 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \xrightarrow{5R_3 \text{ and } 5R_4}$$

$$\left[ \begin{array}{ccc|cc} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \xrightarrow{1/5 \times R_1}$$

$$\left[ \begin{array}{ccc|cc} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -1 & 6/5 & 1/5 & 2/5 \\ 0 & -5 & 6 & 4 & 3 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \xrightarrow{R_2 \times 5}$$

$$\left[ \begin{array}{ccc|cc} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 7 & 8 & 1 \end{array} \right] \xrightarrow{R_3 - R_2}$$

$$\left[ \begin{array}{ccc|cc} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 1 & 8/7 & 4/7 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right] \begin{array}{l} R_3 \leftrightarrow R_4 \\ 1/7 \times R_3 \\ 1/3 \times R_4 \end{array}$$

$$\left[ \begin{array}{ccc|cc} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & -5 & 6 & 1 & 2 \\ 0 & 0 & 1 & 1 & -4/21 \\ 0 & 0 & 0 & 1 & 1/2 \end{array} \right] \xrightarrow{R_2 \times -5}$$

$$\left[ \begin{array}{ccc|cc} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 6 & 1 & 2 \\ 0 & 0 & 1 & 1 & -4/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[ \begin{array}{ccc|cc} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 0 & -5 & 26/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 4/5 & 2/5 & 3/5 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 1 & 1/2 & 3/4 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right] \begin{array}{l} \\ \\ \leftarrow 5/4 \times R_1 \\ \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 126/84 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3/4 \\ 0 & 1 & 0 & 0 & 31/21 \\ 0 & 0 & 1 & 0 & -11/21 \\ 0 & 0 & 0 & 1 & 1/3 \end{array} \right]$$

$$(\alpha, y, z, m) = (3/4, 31/21, -11/21, 1/3)$$

$$\alpha = 3/4$$

$$y = 31/21$$

$$z = -11/21$$

$$m = 1/3$$





Q No 4:- Verify that

$$U(x, t) = \sin(x + 2t)$$

is a solution of the one-dimensional equation.

Solution:-

Given that;

$$U(x, t) = \sin(x + 2t)$$

Differentiate w.r.t  $x$  partially

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \sin(x + 2t)$$

$$\frac{\partial u}{\partial x} = \cos(x + 2t) \frac{\partial}{\partial x} (x + 2t)$$

$$\frac{\partial u}{\partial x} = \cos(x + 2t) (1 + 0)$$

$$\frac{\partial u}{\partial x} = \cos(x + 2t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \cos(x + 2t)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x + 2t) \frac{\partial}{\partial x} (x + 2t)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x + 2t) (1 + 0)$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = -\sin(x+2t)}$$

and

$$u(x,t) = \sin(x+2t)$$

Differentiate w.r.t "t"

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial t} \sin(x+2t)$$

$$\frac{\partial u}{\partial t} = \cos(x+2t) (0+2)$$

$$\frac{\partial u}{\partial t} = 2 \cos(x+2t)$$

$$\frac{\partial^2 u}{\partial t^2} = (2) - \sin(x+2t) (0+2)$$

$$\frac{\partial^2 u}{\partial t^2} = -4 \sin(x+2t)$$

We know that one-dimensional wave equation is,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$-4 \sin(x+2t) = c^2 [-\sin(x+2t)]$$

$$-4 \sin(x+2t) = -c^2 \sin(x+2t)$$

$$-4 \sin(x+2t) + c^2 \sin(x+2t) = 0$$

For the arbitrary constant  $c = \pm 2$   
 $-4 \sin(\alpha + 2t) + 4 \sin(\alpha + 2t) = 0$

$$0 = 0$$

Then it will be verified for the arbitrary  
constant  $c = 2$

