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7279

BEC

DIFFERENTIAL EQUATIONS

1

Question # 1:

(i) $W = \sin(x+ct) + \cos(2x+2ct)$

Given $\frac{\partial w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \longrightarrow$ 1

Now $\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right]$

$= \frac{\partial}{\partial t}(\sin(x+ct)) + \frac{\partial}{\partial t}(\cos(2x+2ct))$

$\frac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct)$

Now $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[c\cos(x+ct) - 2c\sin(2x+2ct)\right]$

$$\boxed{\frac{\partial^2 w}{\partial t^2} = \cancel{\frac{\partial}{\partial t}} -c^2\sin(x+ct) - 4c^2\cos(2x+2ct)}$$

Now $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct) + \cos(2x+2ct)\right]$

$\frac{\partial w}{\partial x} = \cos(x+ct - 2\sin(2x+2ct)$

$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct) - 2\sin(2x+2ct)\right]$

$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$

Eq 1 $\Rightarrow$ $-c^2\sin(x+ct) - 4c^2\cos(2x+2ct) = c^2\left[-\sin(x+ct) - 4\cos(2x+2ct)\right]$

$-c^2\sin(\cancel{x+ct}) - 4c^2\cos(\cancel{2x+2ct}) = -c^2\sin(\cancel{x+ct}) - 4c^2\cos(\cancel{2x+2ct})$

$0 = 0$ satisfied.

(iii) $w = \tan(2x + ct)$

Now $\frac{\partial w}{\partial t} = c\sec^2(2x + ct)$

& $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}(c\sec^2(2x + ct))$

$= c^2 2\sec(2x+ct)\tan(2x+ct)$

Now

$\frac{\partial w}{\partial x} = 2\sec^2(2x + ct)$

$\frac{\partial^2 w}{\partial x^2} = 4\sec^2(2x+ct)\tan(2x+ct)$

$\Rightarrow 4c^2\sec^2\cancel{(2x+ct)}\tan(2x+ct) = 4\sec^2\cancel{(2x+ct)}\tan(2x+ct)$

$0 = 0$ (satisfied)

Question #2:-

$$F(x) = \begin{cases} x & ; -2 < x \le 0 \\ 2x & ; 0 \le x \le \pi \end{cases}$$

to find Fourier co-efficients, a_0, a_n & b_n.

Now.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$a_0 = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi}^{0} + \frac{2}{\pi}\left(\frac{x^2}{2}\right)_{0}^{\pi}$$

$$\Rightarrow a_0 = \frac{-\pi}{2} + \pi$$

$$\boxed{a_0 = \pi/2} \rightarrow (1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (x\cos x\,dx) + \frac{1}{\pi}\int_{0}^{\pi} (2x\cos nx)\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

(4)

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{1-(-1)^n - 2(-1)^n - 2}{n^2}\right]$$

$$= \frac{(-1)^n - 1}{\pi n^2}$$

So $$a_n = \begin{cases} \frac{-2}{\pi n^2} ; & \text{if } n \text{ is odd} \\ 0 ; & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \sin nx \, dx = \frac{1}{\pi}\int_{-\pi}^{0} x \sin nx \, dx + \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi \cos n\pi}{n}\right] + \frac{2}{\pi}\left[-\frac{\pi \cos n\pi}{n}\right] = \frac{-3\cos n\pi}{n}$$

So the Required fourier Series is

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)} + 3\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$$

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Question # 3:-

$$y'' - 4y' + 13y = 8\sin 3x$$

Solution:- characteristic Eq for y_c is;

$$m^2 - 4m + 13 = 0$$

$$\Rightarrow m = \frac{4 \pm \sqrt{16-52}}{2}$$

$$m = 2 \pm 3i \quad , \quad \alpha = 2, \beta = 3$$

So

$$y_c = e^{2x}\{ c_1 \cos 3x + c_2 \sin 3x \}$$

For y_p, Let

$$y_p = \text{Imag} \left(\frac{1}{m^2 - 4m + 13} 8e^{3ix} \right.$$

$$= 8\,\text{Imag} \frac{e^{3ix}}{(3i)^2 - 4(3i) + 13}$$

$$= 8\,\text{Imag} \frac{e^{3ix}}{4 - 12i}$$

Now

$$y_p = \overset{2}{\cancel{8}}\,\text{Imag} \frac{e^{3ix}}{\cancel{4} - \underset{3}{\cancel{12}}i} \times \frac{(1+3i)}{(1+3i)}$$

$$y_p = 2\,\text{Imag} \frac{(1+3i)(e^{3ix})}{10}$$

$$y_p = \frac{2}{10} (\text{Imag} (1+3i)(\cos 3x + i \sin 3x))$$

$$y_p = \frac{2}{10} \{ \sin 3x + 3\cos 3x$$

General Solution $\Rightarrow$ $y = y_c + y_p$

6

$y = y_c + y_p$

$y = c_1 e^{2x}\cos 3x + c_2 e^{2x}\sin 3x + \frac{2}{10}(\sin 3x + 3\cos 3x)$

Now use the initial condition $y(0) = 1$

$y(0) = c_1 e^{(0)}\cos(0) + c_2 e^{(0)}\sin(0) + \frac{2}{10}[\sin(0) + 3\cos(0)]$

$1 = c_1(1) + 0 + 0 + \frac{2}{10}(3(1))$

$c_1 = 1 - \frac{6}{10} = \frac{4}{10} = \frac{2}{5}$ $\Rightarrow$ $\boxed{c_1 = \frac{2}{5}}$

Using another initial cond, $y'(0) = 2$

So $y' = c_1 2e^{2x}\cos 3x + c_1 e^{2x}(-3\sin 3x) + c_2 e^{2x}\sin 3x + c_2 e^{2x}(3\cos 3x)$
$+ \frac{2}{10}(\cos 3x - 3\sin 3x)$

$y'(0) = c_1 2e^{(0)}\cos(0) + c_1 e^{0}(-3\sin 0) + c_2 e^{0}\sin(0) + c_2 e^{0}(3\cos 0)$
$+ \frac{2}{10}(\cos(0) - 3\sin(0))$

$2 = 2c_1 + 0 + 0 + c_2\, 3(1) + \frac{2}{10}(1 - 3(0))$

$2 = 2c_1 + 3c_2 + \frac{2}{10}$

use $c_1 = 2/5$

$2 = 2(2/5) + 3c_2 + 2/10$

$c_2 = \frac{1}{3}(2 - 4/5 - 2/10) = \frac{1}{3}\left(\frac{20 - 8 - 2}{10}\right)$

$\Rightarrow \boxed{c_2 = \frac{1}{3}}$

So the General Solution is

$$y = \frac{2}{5}e^{2x}\cos 3x + \frac{1}{3}e^{2x}\sin 3x + \frac{2}{10}\left[\sin 3x + 3\cos 3x\right]$$

Required Solution.

Question # 4:

$$(D^2 - DD')z = \cos x \cos 2y$$

The auxiliary equation is

$m^2 - m = 0$, $m = 0$, $m = 1$

Hence the complementary function is given by

$$z_c = f_1(y) + f_2(y+x)$$

for the particular integral, we have

$$z_p = \frac{1}{2}\frac{1}{D^2 - DD'}\left[\cos(x+2y) + \cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-DD'}\cos(x+2y) + \frac{1}{D^2-DD'}\cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{-1+2}\cos(x+2y) + \frac{1}{-1-2}\cos(x-2y)\right]$$

$$z_p = \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Hence the complete solution is given by

$$z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Answer.