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Section "A"

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Subject Calculus

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Q1 Find PQ where P is the point in three - dimensional space with coordinates (4,1,3) and the point Q with coordinates (1,2,4). Find the distance b/w P and Q. Further, find the position vector of the point dividing PQ in the ratio 1 : 3.

Solution

$\vec{PQ} = ?$

$P(4,1,3)$, $Q(1,2,4)$

we know that

$\vec{PQ} = \vec{OQ} - \vec{OP}$

Now, position vectors of P and Q are,

$r_1 = \vec{OP} = 4\hat{i} + \hat{j} + 3\hat{k}$

$r_2 = \vec{OQ} = \hat{i} + 2\hat{j} + 4\hat{k}$

Now

$\vec{PQ} = \vec{OQ} - \vec{OP}$

$= (\hat{i} + 2\hat{j} + 4\hat{k}) - (4\hat{i} + \hat{j} + 3\hat{k})$

$= (1-4)\hat{i} + (2-1)\hat{j} + (4-3)\hat{k}$

$\vec{PQ} = -3\hat{i} + \hat{j} + \hat{k}$

Now, we find distance between P and Q.

we know that

$$\text{distance } (d) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$d = \sqrt{(1-4)^2+(2-1)^2+(4-3)^2}$$

$$= \sqrt{(-3)^2+(1)^2+(1)^2}$$

$$= \sqrt{9+1+1}$$

$$d = \sqrt{11}$$

Position vector of Point dividing P and Q in ratio $m_1 : m_2$

$1 : 3$

$$\frac{m_1 r_2 + m_2 r_1}{m_1 + m_2} = \frac{1(\hat{i}+2\hat{j}+4\hat{k})+3(4\hat{i}+\hat{j}+3\hat{k})}{1+3}$$

$$= \frac{\hat{i}+2\hat{j}+4\hat{k}+12\hat{i}+3\hat{j}+9k}{4}$$

$$= \frac{13\hat{i} + 5\hat{j} + 13\hat{k}}{4}$$

$$= \frac{13\hat{i}}{4} + \frac{5\hat{j}}{4} + \frac{13\hat{k}}{4}$$

This is the Position vector of Point which divides $\vec{PQ}$ in the ratio $1 : 3$

Q2 Evaluate

$$\int \frac{4x^3 + 10x + 4}{2x^2 + x}\, dx$$

Solution

$$\int \frac{4x^3 + 10x + 4}{2x^2 + x}\, dx$$

$$\begin{array}{r|l}
 & 2x - 1 \\
\hline
2x^2 + x & \cancel{4x^3} + 10x + 4 \\
 & -\cancel{4x^3} \pm 2x^2 \\
\hline
 & -2x^2 + 10x + 4 \\
 & \mp 2x^2 \mp x \\
\hline
 & 11x + 4
\end{array}$$

$\Rightarrow$ Hence

$$\int \frac{4x^3+10x+4}{2x^2+x} = \int \left(2x-1+\frac{11x+4}{2x^2+x}\right)dx$$

$$= \int (2x-1)\,dx + \int \frac{11x}{2x^2+x}\,dx + \int \frac{4}{2x^2+x}\,dx$$

$$= \int (2x-1)\,dx + \frac{11}{4}\int \frac{4x}{2x^2+x}\,dx + \int \frac{4}{2x^2+x}\,dx$$

$$= \int (2x-1)\,dx + 11/4 \int \frac{4x+1}{2x^2+x}\,dx - 11/4 \int \frac{1}{2x^2+x}\,dx + \int \frac{4}{2x^2+x}\,dx$$

$$= \frac{\cancel{2}x^2}{\cancel{2}} - x + 11/4 \int \frac{4x+1}{2x^2+x}\,dx + (4 - 11/4)\int \frac{dx}{2x^2+x}$$

$$= x^2 - x + 11/4 \ln(2x^2+x) + (5/4)\int \frac{dx}{x(2x-1)} \rightarrow 1$$

Now, let

$$\frac{1}{x(2x+1)} = \frac{A}{x} + B/_{2x-1}$$

$$1 = A(2x+1) + Bx$$

Put $x = 0$

$1 = A(0+1) + B(0) \Rightarrow A = 1$

Put

$x = -\frac{1}{2}$

$1 = A\left(\cancel{2\left(-\frac{1}{2}\right)+1}\right)^{\to 0} + B\left(-\frac{1}{2}\right) \Rightarrow 1 = -\frac{1}{2}B$

$\Rightarrow B = -2$

So

$$\int \frac{dx}{x(2x+1)} = \int \left(\frac{1}{x} - \frac{2}{2x+1} \right) dx = \ln x - \ln(2x+1) + C_1$$

$$\text{Eq } (1) \Rightarrow \int \frac{4x^3 + 10x + 4}{2x^2 + x} dx = x^2 - x - \frac{11}{4} \ln(2x^2 + x) + \left(\frac{5}{4}\right)\left(\ln x - \ln(2x+1)\right) + C_1$$

Ans.

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Q3 Evaluate

(a) $\int_0^2 x^2 e^x \, dx$

Solution

$= \left[x^2e^x - \int e^x \cdot 2x\,dx\right]\Big|_0^2$

$= x^2e^x\Big|_0^2 - \left[2xe^x - \int e^x \cdot 2\,dx\right]_0^2$

$= x^2e^x\Big|_0^2 - 2xe^x\Big|_0^2 + 2\int_0^2 e^x\,dx$

$= x^2e^x\Big|_0^2 - 2xe^x\Big|_0^2 + 2e^x\Big|_0^2$

$= (2^2 \cdot e^2 - 0) - (2(2)e^2 - 0) + 2(e^2 - e^0)$

$= 4e^2 - 4e^2 + 2(e^2 - 1)$

$= 2e^2 - 2$ Ans.

b $\displaystyle\int_1^2 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Let $\sqrt{x} = z$

$\Rightarrow dz = \frac{1}{2}(x)^{-1/2}dx$

$\Rightarrow dz = \frac{1}{2\sqrt{x}}dx$

$\Rightarrow dx = 2\sqrt{x}\cdot dz$

$\Rightarrow \boxed{dx = 2z\,dz}$

$$= \int_1^2 \frac{\sin z}{z}\cdot 2z\,dz$$

$$= 2\int_1^2 \sin z\,dz$$

$$= 2(-\cos z)\Big|_1^2$$

$$= -2\cos\sqrt{x}\Big|_1^2$$

$$= -2\left(\cos\sqrt{2} - \cos\sqrt{1}\right)$$

$$= 2(\cos\sqrt{2} - \cos(1))$$

$$= -2(0.9996 - 0.9998)$$

$$= -2(-0.0002)$$

$$= (-0.0004) \text{ Ans.}$$

Q4 Verify that

$$u(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

Satisfies the three-dimensional laplace's equation.

<u>Sol</u>

To verify that is satisfies the three dimensional Equation.

<u>Proof</u> :- The daplacion three dimensional equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 z}{\partial z^2} = 0$$

Now

$$u = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$$

$$\Rightarrow \frac{\partial u}{\partial x} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}(\cancel{2}x)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -(x^2+y^2+z^2)^{-3/2}(1) + x \cdot \frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}(\cancel{2}x)$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}}$$

Also

$$\frac{\partial u}{\partial y} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}(\cancel{2}y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -(x^2+y^2+z^2)^{-3/2}(1) + y\left(\frac{3}{\cancel{2}}\right)(x^2+y^2+z^2)^{-5/2}(\cancel{2}y)$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial y^2} = -(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2}}$$

Similarly

$$\boxed{\frac{\partial^2 u}{\partial z} = -(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2}}$$

Now

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \bullet$$

$$-(x^2+y^2+z)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2}$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2} + 3y^2(x^2+y^2+z^2)^{-5/2} + 3z^2(x^2+y^2+z^2)^{-5/2}$$

$$= -3(x^2+y^2+z^2)^{-3/2}\left[1 - x^2(x^2+y^2+z^2)^{-1} - y^2(x^2+y^2+z^2)^{-1} - z^2(x^2+y^2+z^2)^{-1}\right]$$

$$= -3(x^2+y^2+z^2)^{-3/2}\left[1 - \frac{x^2}{x^2+y^2+z^2} - \frac{y^2}{x^2+y^2+z^2} - \frac{z^2}{x^2+y^2+z^2}\right]$$

$$= -3(x^2+y^2+z^2)^{-3/2}\left[\frac{\cancel{x^2}+\cancel{y^2}+\cancel{z^2}-\cancel{x^2}-\cancel{y^2}-\cancel{z^2}}{x^2+y^2+z^2}\right]$$

$$= -3(x^2+y^2+z^2)^{-3/2}(0)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Hence Laplace equation is satisfied.