SHAH HASSAN

Sec – "B"

ID – 7978

Subject – Differential Equation

To;

Ma'am. Shomaila Mazhar.

Q-01 (Part i)

Solution:

$$W = \sin(x+ct) + \cos(2x+2ct)$$

$$\frac{\partial w}{\partial t} = \cos(x+ct) + c - \sin(2x+2ct) + 2c$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) + c^2 - \cos(2x+2ct) + 4c^2 \ldots \text{ — (1)}$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - \sin(2x+2ct) + 2$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$= [-\sin(x+ct) - 4\cos(x+2ct)]$$

$$\frac{\partial^2 w}{\partial t^2} = c^2 [-\sin(x+ct) - 4\cos(2x+2ct)]$$

$$\Rightarrow -c^2 + \frac{\partial w}{\partial x^2}$$

<u>Part (ii)</u> $\quad w = \tan x \,(2x + 2ct)$

<u>Solution:</u>

$$\frac{dw}{dx} = \frac{1+2}{[1+(2x+2ct)^2]}$$

$$= \frac{2}{1+4x^2+8xct+4c^2+2}$$

$$\frac{d^2w}{dx^2} = -2\left(1+4x^2+8xct+4c^2+2\right)^2 + (8x+8ct)$$

$$\frac{dw}{dt} = \frac{1+(2c)}{1+(2x+2ct)^2}$$

$$= \frac{2c}{[1+4x^2+4c^2t^2+8xct]^2}$$

$$\frac{d^2w}{dt^2} = \frac{2c(1)(4c^2+2t+txc}{[1+4x^2+4c^2t^2+8xct]^2}$$

$$\frac{d^2w}{dt^2} = \frac{2c^2[4+8ct-8x]}{[1+4x^2+4c^2t^2-8xct]^2}$$

$$\frac{d^2w}{dt^2} = \frac{2c^2[8x+8ct]}{1+4x^2+4c^2t^2+4c^2t^2}$$

$$\frac{d^2w}{dt^2} = \frac{d^2w}{dx^2} + c^2.$$

Q - No (02)

Given funtion is

$$F(x) = \begin{cases} x ; & -\pi < x \le 0 \\ 2x ; & 0 \le x \le \pi . \end{cases}$$

We have to find the Foriers co-efficient a_0, a_n & b_n.

Now;

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{\pi^2}{2}\right] + \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$\boxed{a_0 = \frac{-\pi}{2} + \pi = \frac{\pi}{2}} \quad -(1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(x\cos nx\,dx) + \frac{1}{\pi}\int_{0}^{\pi}(2x\cos nx)\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin x}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{\sin x}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^2 - 1}{\pi n^2}$$

So,

$$a_n = \begin{cases} \frac{-2}{\pi n^2}\,; & \text{if } n \text{ is odd} \\ 0\,; & \text{if } n \text{ is even} \end{cases} \quad -(2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$$

$$+ \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{x^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(-\frac{\cos nx^2}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

3

$$\boxed{b_n = \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] = -\frac{3\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}}$$

So the required Fourier series is,

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2x-1)}{(2x-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

Q-03:

Solution:

$$y'' - 4y' + 13y = 8\sin 3x \quad —(1)$$

Associated homogenous eq of "1" is

$$y'' - 4y' + 13y = 0 \quad —(2)$$

Change eq "2" into Auxiliary eq;

put $y = m$ in eq (2)

$$m^2 - 4m + 13 = 0$$

Use quardratic formula.

$$a = 1, \quad b = -4, \quad c = 13$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$m = \frac{4 \pm \sqrt{16-52}}{2}$$

$$m = \frac{4 \pm \sqrt{-36}}{2}$$

$$m = \frac{4 \pm 6i}{2}$$

$$m = 2 \pm 3i$$

$$\boxed{m_1 = 2 + 3i}$$

$$\boxed{m_2 = 2 - 3i}$$

$$y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) \quad \text{— (A)}$$

let, $y_p = A \cos 3x + B \sin 3x$ — (#)

Diff w.r.t 'x'

$$y_p' = -3A \sin 3x + 3B \cos 3x$$

Again diff w.r.t 'x'

$$y_p'' = -9A\cos 3x - 9B\sin 3x$$

Put in eq (1)

$$\Rightarrow (-9A\cos 3x - 9B\sin 3x) - 4(-3A\sin 3x +$$

$$3B\cos 3x) + B(A\cos 3x + B\sin 3x - B\sin 3x$$

$$\Rightarrow -9A\cos 3x - 12B\cos 3x + 13A\cos 3x - 9B\sin 3x$$

$$+ 12A\sin 3x + 13B\sin 3x = 8\sin 3x$$

$$\Rightarrow (-9A - 12B + 13A)\cos 3x + (-9B + 12A + 13B)\sin 3x = 8\sin 3x$$

$$\Rightarrow (4A - 12B)\cos 3x + (4B + 12A)\sin 3x = 8\sin 3x$$

Comparing co-efficients

$$\sin 3x \Rightarrow 4B + 12A = 8 \quad - (a)$$

$$\cos 3x \Rightarrow 4A - 12B = 0 \Rightarrow 4A = 12B$$

$$A = 3B \quad - (b)$$

Put b in a

$$4B + 12(3B) = 8$$

$$4B = 8$$

$$\boxed{B = \frac{1}{5}} - (i)$$

Put 'i in 'b'

$$A = \frac{3}{5} \quad -(d)$$

Put (i) & (d) in (x)

$$y_p = \frac{3}{5}\cos 3x + \frac{1}{5}\sin x - (B)$$

The G. Solution is

$$y = y_c + y_p$$

$$y = e^{2x}\left(c_1 \cos 3x + c_2 \sin 3x\right) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x - (C)$$

Now we need to find the values of c_1 & c_2

Put $x = 0$ & $y = 1$ in (C)

$$1 = e^{2(0)}\left(c_1 \cos 3(0) + c_2 \sin 3(0)\right) + \frac{3}{5}\cos 3(0)$$
$$+ \frac{1}{5}\sin 3(0)$$

$$1 = \left(C_1(1) + C_2(0)\right) + \frac{3}{5}(1) + \frac{1}{5}(0)$$

$$1 = C_1 + \frac{3}{5}$$

$$C_1 = 1 - \frac{3}{5}$$

$$\boxed{C_1 = \frac{2}{5}} \text{ — } ☆☆$$

Diff ① w.r.t 'x'

$$y' = C_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + C_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x) - \frac{6}{5}\sin x + \frac{3}{5}\cos 3x \text{ — } ⓓ$$

Put $y' = 2,\ x = 0$ in ⓓ

$$y' = C_1(2e^{2x}\cos 3x - 3x^{2x}\sin 3x)$$
$$+ C_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x)$$
$$- \frac{6}{5}\sin 3x + \frac{3}{5}\cos 3x$$

Put $y' = 2,\ x = 0$

$$2 = C_1(2e^{2(0)}\cos 3(0) - 3e^{2(0)}\sin 3(0))$$
$$C_2(2e^{2(0)}\sin 3(0) + 3e^{2(0)}\cos 3(0))$$
$$- \frac{6}{5}\sin 3(0) + \frac{3}{5}\cos 3(0)$$

$$2 = C_1(2) + C_2(3) - 0 + \frac{3}{5}$$

$$2 = 2C_1 + 3C_2 + \frac{3}{5}$$

Put, $C_1 = \frac{2}{5}$

$$2 = 2\left(\frac{2}{5}\right) + 3C_2 + \frac{3}{5}$$

$$2 = \frac{7}{5} + 3C_2$$

$$\boxed{C_2 = \frac{3}{15}} \quad ★★★$$

Put ★★ & ★★★ in (C)

$$y = e^{2x}\left(\frac{2}{5}\cos 3x + \frac{3}{15}\sin 3x\right) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x$$

$$\boxed{y = \frac{2}{5}e^{2x}\cos 3x + \frac{3}{15}e^{2x}\sin 3x + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x}$$

is Required General Solution.

Q-No-04

$$(D^2 - DD')z = \cos x \cos 2y$$

Solution: The given PDE can be rewrite as;

$$D(D-D')u = \cos x \cos 2y$$

In CF is given by

$$CF = \phi_1(y) + \phi_2(y+x)$$

while Its PI is given by;

$$PI = \frac{1}{(D^2-DD')} \cdot \frac{1}{2}\left[\cos(x+2y) + \cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{-1+2}\cos(x+2y) + \frac{1}{-1-2}\cos(x-2y)\right]$$

$$= \frac{1}{2}\cos(x+2y - \frac{1}{6}\cos(x-2y)$$

Hence the complete solution of given PDE is

$$u = \phi_1(y) + \phi_2(x+y) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$ Ans.