1

Final Term Paper BS : SE-II

Name :- AFAQ AHMAD SHAH

ID # :- 15715

Course :- Linear Algebra

Q#01 To determine whether the system is consistent or not;

$$x_1 - (3rd\text{-}ID)x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$5x_1 - 5x_3 = 10.$$

Solution :-

As Given;

$$x_1 - (3rd\text{-}ID)x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$5x_1 - 5x_3 = 10$$

where; 3rd-ID # 7.

$\therefore \Rightarrow$

$$x_1 - 7x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$5x_1 - 5x_3 = 10$$

Isolate x_2 : $2x_2 - 8x_3 = 8$;

$$2x_2 - 8x_3 = 8$$
$$\Rightarrow 2x_2 = 8 + 8x_3$$
$$\Rightarrow x_2 = \frac{8 + 8x_3}{2}$$

2

$\Rightarrow \quad x_2 = \frac{8}{2} + \frac{8x_3}{2}$

$\Rightarrow \quad x_2 = 4 + 4x_3$

OR $\Rightarrow \quad x_2 = 4x_3 + 4$

$\Rightarrow \quad \boxed{x_2 = 4\,(x_3 + 1).} \longrightarrow (1)$

Substitute Eq 1) in : $x_1 - 7x_2 + x_3 = 0$;

$\therefore \quad x_1 - 7 \cdot 4\,(x_3 + 1) + x_3 = 0$

$\Rightarrow \quad x_1 - 28\,(x_3 + 1) + x_3 = 0$

$\Rightarrow \quad x_1 - 28x_3 - 28 + x_3 = 0$

OR $\quad -28x_3 + x_3 + x_1 - 28 = 0$

$\Rightarrow \quad \boxed{-27x_3 + x_1 - 28 = 0.} \longrightarrow (2)$

Now, Isolate x_1 : $5x_1 - 5x_3 = 10$ and put the value in Eq 2) ;

$\therefore \quad 5x_1 - 5x_3 = 10$

$\Rightarrow \quad 5x_1 = 10 + 5x_3$

$\Rightarrow \quad x_1 = (10 + 5x_3)/5$

$\Rightarrow \quad x_1 = 10/5 + 5x_3/5$

$\Rightarrow \quad \boxed{x_1 = 2 + x_3} \longrightarrow$ put in Eq 2)

$\therefore$ Eq 2)

$\Rightarrow \quad -27x_3 + (2 + x_3) - 28 = 0$

$\Rightarrow \quad -27x_3 + x_3 + 2 - 28 = 0$

$\Rightarrow \quad -26x_3 - 26 = 0$

(P.T.O)

3

Isolate: x_3 : $-26x_3 - 26 = 0$;

$\therefore \quad -26x_3 - 26 = 0$

$\Rightarrow -26x_3 = +26$

$\Rightarrow x_3 = \frac{26}{-26}$

$\Rightarrow \boxed{x_3 = -1} \longrightarrow$ #)

put Eq #) in Eq ☆), therefore;

Eq ☆) $\Rightarrow \quad x_1 = 2 + (-1)$

$\Rightarrow x_1 = 2 - 1$

$\Rightarrow \boxed{x_1 = 1} \longrightarrow$ ☆')

Now, for $x_2 = 4(x_3 + 1)$;

substitute $x_1 = 1$, $x_3 = -1$

in Eq 1)

$\therefore$ Eq 1) $\Rightarrow \quad x_2 = 4(-1+1)$

$\Rightarrow x_2 = 4(0)$

$\Rightarrow \boxed{x_2 = 0} \longrightarrow$ #')

Finally: from Eq ☆'), #) and #')

we have;

$x_1 = 1$, $x_2 = 0$, $x_3 = -1$

Hence, the Given system of Solutions is Consistent.

4

Q#02 Inverse of $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 4^{th}\text{-ID} \\ 5 & -2 & 7 \end{bmatrix}$

by adjoint method;

Sol:- As Given;

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 1 \\ 5 & -2 & 7 \end{bmatrix}$$, where; 4^{th}-ID # 1

I :- Calculate the Determinant;

$$\therefore |A| = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 1 \\ 5 & -2 & 7 \end{vmatrix}$$, Expand by R_1

$$\Rightarrow |A| = 3|-7+2| - 4|14-5| + 5|-4+5|$$

$$\Rightarrow |A| = 3(-5) - 4(9) + 5(1)$$

$$\Rightarrow |A| = -15 - 36 + 5$$

$$\Rightarrow \boxed{|A| = -46} \rightarrow \star)$$

Now;

(II): Calculate the Adjoint;

(P.T.O)

5

$$\therefore \quad A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 1 \\ 5 & -2 & 7 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} -1\times 7 - 1\times(-2) & 2\times 7 - 1\times 5 & 2\times(-2) - (-1)\times 5 \\ 4\times 7 - 5\times(-2) & 3\times 7 - 5\times 5 & 3\times(-2) - 4\times 5 \\ 4\times 1 - 5\times(-1) & 3\times 1 - 5\times 2 & 3\times(-1) - 4\times 2 \end{bmatrix}^T$$

$$\Rightarrow \text{Adj } A = \begin{bmatrix} -7+2 & 14-5 & -4+5 \\ 28+10 & 21-25 & -6-20 \\ 4+5 & 3-10 & -3-8 \end{bmatrix}^T$$

$$\Rightarrow \text{Adj } A = \begin{bmatrix} (+)-5 & (-)9 & (+)1 \\ (-)38 & (+)-4 & (-)-26 \\ (+)9 & (-)-7 & (+)-11 \end{bmatrix}^T$$

$$\Rightarrow \text{Adj } A = \begin{bmatrix} -5 & -9 & +1 \\ -38 & -4 & 26 \\ 9 & 7 & -11 \end{bmatrix}^T$$

$$\Rightarrow \text{Adj } A = \begin{bmatrix} -5 & -38 & 9 \\ -9 & -4 & 7 \\ 1 & 26 & -11 \end{bmatrix}$$

$\therefore$ Inverse of $A = \frac{1}{|A|} \cdot \text{adj } A$

$$\Rightarrow \text{Inverse of } A = \frac{1}{-46}\begin{bmatrix} -5 & -38 & 9 \\ -9 & -4 & 7 \\ 1 & 26 & -11 \end{bmatrix}$$

Result.

Q#03: Linear System by Guass-Jordan Method;

$$2x + 2y + 4z = 18$$
$$x + 3y + 2z = 13$$
$$3x + 2y - 3z = 14$$

Sol:

The above given linear system can be written in matrix form as;

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 18 \\ 1 & 3 & 2 & 13 \\ 3 & 2 & -3 & 14 \end{array}\right]$$

$\downarrow$

$\therefore \begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 2 & -3 \end{bmatrix}$, solving by Guass Jordan,

I: Echelon form;

P.T.O

7

$$R_1 \longleftrightarrow R_3$$

$$\sim \begin{bmatrix} 3 & 2 & -3 \\ 1 & 3 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$R_2 - \tfrac{1}{3}R_1 \longrightarrow R_1$$

$$\sim \begin{bmatrix} 3 & 2 & -3 \\ 0 & 7/3 & 3 \\ 2 & 2 & 4 \end{bmatrix}$$

$$R_3 - \tfrac{2}{3}R_1 \longrightarrow R_3$$

$$\sim \begin{bmatrix} 3 & 2 & -3 \\ 0 & 7/3 & 3 \\ 0 & 2/3 & 6 \end{bmatrix}$$

$$R_3 - \tfrac{2}{7}R_2 \longrightarrow R_3$$

$$\sim \begin{bmatrix} 3 & 2 & -3 \\ 0 & 7/3 & 3 \\ 0 & 0 & 36/7 \end{bmatrix}.$$

Now, further Reduce the matrix to Reduced Row Echelon form;

8

$$\therefore \begin{bmatrix} 3 & 2 & -3 \\ 0 & 7/3 & 3 \\ 0 & 0 & 36/7 \end{bmatrix}$$

$$\tfrac{7}{36} \cdot R_3 \longrightarrow R_3$$

$$\sim \begin{bmatrix} 3 & 2 & -3 \\ 0 & 7/3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - 3R_3 \longrightarrow R_2$$

$$\sim \begin{bmatrix} 3 & 2 & -3 \\ 0 & 7/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 + 3 \cdot R_3 \longrightarrow R_1$$

$$\sim \begin{bmatrix} 3 & 2 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tfrac{3}{7} \cdot R_2 \longrightarrow R_2$$

$$\sim \begin{bmatrix} 3 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(P.T.O)

9

$R_1 - 2 \cdot R_2 \rightarrow R_1$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{1}{3} \cdot R_1 \longrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Hence, the linear system solved by Guass-Jordan method Gives us;

$x = 1 \ , \ y = 1 \ , \ z = 1$

= Result

Q#05 To describe the sol set if the Given homogeneous system has non-trivial Solution;

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$-3x_1 - 25x_2 + 4x_3 = 0$$
$$6x_1 + x_2 - 8x_3 = 0$$

10

Sol:-

As Given;

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$-3x_1 - 25x_2 + 4x_3 = 0$$
$$6x_1 + x_2 - 8x_3 = 0$$

In Matrix form;

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -25 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right]$$

we will solve the solution of these equations by Gaussian Elimination;

∴ Reduce matrix to Row Echelon form;

i.e: $$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -25 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

$$R_1 \longleftrightarrow R_3$$

$$\sim \begin{bmatrix} 6 & 1 & -8 & 0 \\ -3 & -25 & 4 & 0 \\ 3 & 5 & -4 & 0 \end{bmatrix}$$

$$R_2 + \tfrac{1}{2} \cdot R_1 \longrightarrow R_2$$

11

$$\sim \begin{bmatrix} 6 & 1 & -8 & 0 \\ 0 & -49/2 & 0 & 0 \\ 3 & 5 & -4 & 0 \end{bmatrix}$$

$$R_3 - \tfrac{1}{2} \cdot R_1 \longrightarrow R_3$$

$$\sim \begin{bmatrix} 6 & 1 & -8 & 0 \\ 0 & -49/2 & 0 & 0 \\ 0 & 9/2 & 0 & 0 \end{bmatrix}$$

$$R_3 + \tfrac{9}{49} \cdot R_2 \rightarrow R_3$$

$$\sim \begin{bmatrix} 6 & 1 & -8 & 0 \\ 0 & -49/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now,

Reduce Matrix to Reduced Row Echelon Form;

$$\begin{bmatrix} 6 & 1 & -8 & 0 \\ 0 & -49/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\frac{2}{49} \cdot R_2 \longrightarrow R_2$$

12

$$\sim \begin{bmatrix} 6 & 1 & -8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 - 1 \cdot R_2 \rightarrow R_1$$

$$\sim \begin{bmatrix} 6 & 0 & -8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tfrac{1}{6} \cdot R_1 \rightarrow R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Zero Row in Reduced Matrix indicates infinite solutions.

i.e:

$$x_1 - \tfrac{4}{3} x_3 = 0 \Rightarrow x_1 = 0$$

$$\boxed{\begin{aligned} x_2 &= 0 \\ x_3 &= 0 \end{aligned}}$$

substitute

Also;

As we see, the Rank of Matrix is 2 and the number of unknowns in system are 3.

so; $2 < 3$. According to condition, the system has non-trivial solutions.

Q#04 To show that the given matrix is diagonalizeable?

$$\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

Sol:-

Let;

$$A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

∴ A matrix A can be diagonalized if there exists an invertible matrix P and a diagonable matrix D i-e

$$A = PDP^{-1}$$

∴ Eigenvalues For;

$$A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

• Let A be a (square → matrix), η a vector and λ a scalar, that satisfy $A\eta = \lambda\eta$, then λ is called the Eigen value associated with the eigen-vector η of A.

14 . The eigenvalues of A are the roots of the characteristic equation ; $\det(A-\lambda I)=0$

i.e: $\det\left(\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$

$$= \det\left(\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right)$$

$$= \det \begin{bmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{bmatrix}$$

$$= \begin{vmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{vmatrix}$$

$$= (4-\lambda)\left|(3-\lambda)*(1-\lambda)-8\right|$$

$$-(2)\left|(-5)(1-\lambda)-(2)(-2)\right|$$

$$+(-2)\left|(-5)(4)-(3-\lambda)(-2)\right|$$

P.T.O

15

$(3-\lambda)(1-\lambda)-8$

$= 3-3\lambda-\lambda+\lambda^2-8$

$= \underline{\underline{(\lambda^2-4\lambda-5)}}$

$(-5)(1-\lambda)+4$

$= -5+5\lambda+4$

$= \underline{\underline{(5\lambda-1)}}$

$(-20)(-6+2\lambda)$

$= -20+6-2\lambda$

$= (-2\lambda-14)$

$=$

$= (4-\lambda)(\lambda^2-4\lambda-5)-2(5\lambda-1)-2(-2\lambda-14)$

$= 4\lambda^2-16\lambda-20-\lambda^3+4\lambda^2+5\lambda-10\lambda+2+4\lambda+28$

$= -\lambda^3+8\lambda^2-17\lambda+10$

$= -(\lambda-1)(\lambda-2)(\lambda-5)$

Now applying/using zero factor principle i-e;

$ab=0$, then $a=0, b=0$.

$\therefore$

$$\left.\begin{array}{lll} \lambda-1=0 & \Rightarrow & \lambda=1 \\ \lambda-2=0 & \Rightarrow & \lambda=2 \\ \lambda-5=0 & \Rightarrow & \lambda=5 \end{array}\right] \text{solutions.}$$

Therefore;

Eigenvalues are;

$\lambda=1, \lambda=2, \lambda=5$

So, Diagonal Matrix "D" is composed of Eigen values

i-e;

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Now, the Eigenvectors for

$$\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

16

∴ let A be a square matrix, η a vector & λ a scalar that satisfy $A\eta = \lambda\eta$.
Then; η is an eigenvector of A and λ is the eigen-value associated with it.

1: Eigenvectors for $\lambda = \underline{\underline{1}}$:-

As; $(A - \lambda I) = 0$

∴
$$A - \lambda I = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{bmatrix}.$$

Now;

$\lambda = 1$

$$\therefore A - \lambda I = \begin{bmatrix} 4-1 & 2 & -2 \\ -5 & 3-1 & 2 \\ -2 & 4 & 1-1 \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}$$

Now; To Solve;

P.T.O

17

$$\begin{bmatrix} +3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the matrix:

i.e,

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}$$

Reduce by Echelon Method;

$$R_1 \longleftrightarrow R_2$$

$$\sim \begin{bmatrix} -5 & 2 & -2 \\ 3 & 2 & -2 \\ -2 & 4 & 0 \end{bmatrix}$$

$$R_2 + 3/5 \cdot R_1 \longrightarrow R_2$$

$$\sim \begin{bmatrix} -5 & 2 & -2 \\ 0 & 16/5 & -4/5 \\ -2 & 4 & 0 \end{bmatrix}$$

$$R_3 - 2/5\, R_1 \longrightarrow R_3$$

$$\sim \begin{bmatrix} -5 & 2 & -2 \\ 0 & 16/5 & -4/5 \\ 0 & 16/5 & -4/5 \end{bmatrix}$$

18

$R_3 - 1R_2 \to R_3$

$$\sim \begin{bmatrix} -5 & 2 & 2 \\ 0 & 16/5 & -4/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, further Reduce by Reduced Row Echelon form;

$\therefore \quad \frac{5}{16} \cdot R_2 \to R_2$

$$\sim \begin{bmatrix} -5 & 2 & 2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 2R_2 \to R_1$

$$\sim \begin{bmatrix} -5 & 0 & 5/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$-\frac{1}{5} R_1 \to R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

Now,

the system associated with

19 eigenvalue ; $\lambda = 1$.

$$(A - 1I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduces to

$\Rightarrow \quad 1x - \frac{1}{2}z = 0 \quad , \Rightarrow \quad x = \frac{1}{2}z$

$\quad 1y - 1/4\, z = 0 \quad , \Rightarrow \quad y = 1/4\, z$

Plug into $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, i-e

$$\eta = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2\, z \\ 1/4\, z \\ z \end{bmatrix}, \quad z \neq 0. \text{ let } z = 4$$

$$\Rightarrow \eta = \begin{bmatrix} 4/2 \\ 4/4 \\ 4 \end{bmatrix}, \Rightarrow \eta = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

2: Eigenvectors for $\lambda = 2$

$$\therefore (A - \lambda I) = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A - 1I = \begin{bmatrix} 4-2 & 2 & -2 \\ -5 & 3-2 & 2 \\ -2 & 4 & 1-2 \end{bmatrix}$$

20

$$\Rightarrow A - 1I = \begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}$$

Now, to solve;

$$\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the matrix: $\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}$

by Echelon Form;

Swap : $R_1 \longleftrightarrow R_2$

$$\sim \begin{bmatrix} -5 & 1 & 2 \\ 2 & 2 & -2 \\ -2 & 4 & -1 \end{bmatrix}$$

$$R_2 + \frac{2}{5} \cdot R_1 \longrightarrow R_2$$

$$\sim \begin{bmatrix} -5 & 1 & 2 \\ 0 & \frac{12}{5} & -\frac{6}{5} \\ -2 & 4 & -1 \end{bmatrix}$$

$$R_3 - \frac{2}{5} \cdot R_1 \longrightarrow R_3$$

P.T.O

$$\sim \begin{array}{cc} \begin{bmatrix} -5 & 1 & 2 \\ 0 & \frac{12}{5} & -\frac{6}{5} \\ 0 & \frac{18}{5} & -\frac{9}{5} \end{bmatrix} & \begin{matrix} \\ R_2 \\ R_3 \end{matrix} \end{array}$$

$$R_2 \longleftrightarrow R_3$$

$$\sim \begin{bmatrix} -5 & 1 & 2 \\ 0 & 18/5 & -9/5 \\ 0 & 12/5 & -6/5 \end{bmatrix}$$

$$R_3 - \frac{2}{3} R_2 \longrightarrow R_3$$

$$\sim \begin{bmatrix} -5 & 1 & 2 \\ 0 & 18/5 & -9/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Now,

Further Reduce the above matrix by Reduced Row Echelon form,

i.e,

$$\frac{5}{18} \cdot R_2 \longrightarrow R_2$$

$$\sim \begin{bmatrix} -5 & 1 & 2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

22

$R_1 - 1 \cdot R_2 \longrightarrow R_1$

$$\sim \begin{bmatrix} -5 & 0 & 5/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$-\frac{1}{5} \cdot R_1 \longrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

now,
the system associated with the eigenvalue; $\lambda = 2$

$$(A - 2I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The above matrix reduces to the following system of equations;

$x - \frac{1}{2}z = 0 \quad , \Rightarrow x = \frac{1}{2}z$

$y - \frac{1}{2}z = 0 \quad , \Rightarrow y = \frac{1}{2}z$

plug into $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

i-e;

$$\eta = \begin{bmatrix} \frac{1}{2} z \\ \frac{1}{2} z \\ z \end{bmatrix}, \quad z \neq 0, \quad \text{let } z = 2$$

$$\Rightarrow \eta = \begin{bmatrix} 2/2 \\ 2/2 \\ 2 \end{bmatrix}$$

$$\Rightarrow \eta = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \longrightarrow 2$$

Now;

3) Eigenvectors for $\lambda = 5$,

solve; $(A - \lambda I)$:

$$\Rightarrow \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow A - 5I = \begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix}$$

24

Now, reduce the matrix by Row Echelon Form;

i.e: $\begin{bmatrix} -5 & -2 & 2 \\ -1 & 2 & -2 \\ -2 & 4 & -4 \end{bmatrix}$, $R_1 \leftrightarrow R_2$

$$R_2 - \frac{1}{5} \cdot R_1 \longrightarrow R_2$$

$$\sim \begin{bmatrix} -5 & -2 & 2 \\ 0 & 12/5 & -12/5 \\ -2 & 4 & -4 \end{bmatrix}$$

$$R_3 - \frac{2}{5} \cdot R_1 \longrightarrow R_3$$

$$\sim \begin{bmatrix} -5 & -2 & 2 \\ 0 & 12/5 & -\frac{12}{5} \\ 0 & 24/5 & -\frac{24}{5} \end{bmatrix}$$

$$R_2 \longleftrightarrow R_3$$

$$\sim \begin{bmatrix} -5 & -2 & 2 \\ 0 & 24/5 & -24/5 \\ 0 & 12/5 & -12/5 \end{bmatrix}$$

$$R_3 - \frac{1}{2} \cdot R_2 \longrightarrow R_3$$

25

$$\sim \begin{bmatrix} -5 & -2 & 2 \\ 0 & 24/5 & -24/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Now,

Further Reduce the above matrix by Reduced Row Echelon form;

$$\frac{5}{24} \cdot R_2 \to R_2$$

$$\sim \begin{bmatrix} -5 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 + 2 \cdot R_2 \longrightarrow R_1$$

$$\sim \begin{bmatrix} -5 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-1/5 \cdot R_1 \longrightarrow R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

26

The system associated with eigenvalue $\lambda = 5$

$$(A - 5I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This reduces to;

$$x = 0 \quad , \Rightarrow \quad x = 0$$

$$y - z = 0 \quad , \Rightarrow \quad y = z$$

Plug into; $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

i.e.,

$$\eta = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ z \\ z \end{bmatrix}$$

$$z \neq 0$$

Let $z = 1$

$\therefore$ $\eta = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, The eigenvectors corresponding to the eigenvalues in D compose the columns of P;

$$P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Now;

$$P^{-1} = ?$$

$$\therefore \quad P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}^{-1}$$

$\therefore$ Augmented with a 3×3 identity matrix

i.e:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Reduce by Echelon form

$$R_1 \longleftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$R_2 - \frac{1}{4} R_1 \longrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1/2 & 3/4 & 0 & 1 & -1/4 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$R_3 - \tfrac{1}{2} \cdot R_1 \longrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{3}{4} & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \end{array}\right]$$

Now, further Reduce by Reduced Row Echelon form;

$$-2R_3 \longrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{3}{4} & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$R_2 - \frac{3}{4} R_3 \longrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 2 & 1 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$R_1 - 1R_3 \longrightarrow R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 2 & 0 & 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$R_1 - 2R_2 \longrightarrow R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 3 & 2 & -2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\tfrac{1}{4} R_1 \longrightarrow R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 3 & 2 & -2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}.$$

Now,

$$PDP^{-1} = ?$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

30

$$= \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 5 \\ 4 & 4 & 5 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

$$PDP^{-1} = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$, Diagonalized

$\therefore \quad A = PDP^{-1} \longrightarrow$ Result

Q#06 Reduce matrix to Normal form and Rank of matrix?

Sol:

As;
$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 0 \end{bmatrix}$$

By Row operation;

$R_2 \leftarrow R_2 - 3R_1$; $R_3 - R_1 \rightarrow R_3$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

31

$3R_3 - R_2 \longrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now; column operation

- $C_2 - 3C_1 \longrightarrow C_2$, $C_4 \leftrightarrow C_2$
- $C_3 - 4C_1 \longrightarrow C_3$, $\frac{C_2}{-6} \longrightarrow C_2$
- $C_4 - 3C_1 \longrightarrow C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{unit vector.}$$

Rank:-

$$\text{Rank} \begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 0 \end{pmatrix}$$

by

Reduce matrix to reduced Row Echelon form.

· Row Echelon form:

· $R_1 \leftrightarrow R_2$

· $R_2 - \frac{1}{3} R_1 \to R_2$

· $R_3 - \frac{1}{3} \cdot R_1 \to R_3$

$$\sim \begin{bmatrix} 3 & 9 & 12 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Now:

Reduce Echelon form,

· $-1R_3 \to R_3$

· $R_2 - 2R_3 \to R_2$

· $R_1 - 3R_3 \to R_1$

· $R_2 \leftrightarrow R_3$

· $R_1 - \frac{1}{3} \cdot R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boxed{\text{Rank} = 2}$$

(Rank => no. of non zero rows)

Result.