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Paper: Difficerential Equations:

QNo1: The wave equation:

we generally visit beach.......

.... $w = \tan(2x+ct$.

(i):- $W = \sin(x+ct) + \cos(2x+2ct)$.

Given:

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \longrightarrow ①.$$

Now:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$= \frac{\partial}{\partial t}\left(\sin(x+ct)\right) + \frac{\partial}{\partial t}\left(\cos(2x+2ct)\right).$$

$$\frac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct).$$

Now.

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[c\cos(x+ct) - 2c\sin(2x+2ct)\right]$$

$$\frac{\partial^2 w}{\partial t^2} = -c^2\sin(x+ct) - 4c^2\cos(2x+2ct).$$

Now:

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - 2\sin(2x+2ct).$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct) - 2(\sin(2x+2ct)\right]$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct).$$

$$-c^2\sin(x+ct) - 4c^2\cos(2x+2ct) = c^2\left[-\sin(x+ct-4\cos(2x+2ct)\right].$$

$$-c^2\sin(x+ct) - 4c^2\cos(2x+2ct = c^2\left[-\sin(x+ct-4\cos(2x+2ct)\right].$$

$$-c^2\cancel{\sin}(x+ct) - 4c^2\cos(\cancel{2x}+2ct = c^2\cancel{\sin}(x+ct - 4c^2\cos\cancel{(2x+2ct)}]$$

$$0 = 0 \text{ (satisfied)}.$$

ii): $W = \tan(2x+ct)$.

Now: $\frac{\partial w}{\partial t} = c\sec^2(2x+ct)$.

&

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(c\sec^2(2x+ct)\right).$$

$$c2^2 \cdot 2\sec^2(2x+ct)\tan(2x+ct).$$

Now

$$\frac{\partial w}{\partial x} = 2\sec^2(2x+ct).$$

$$\frac{\partial^2 w}{\partial t^2} = 4\sec^2(2x+ct)\tan(2x+ct).$$

$$\Rightarrow 4c^2\sec^2(2x+ct)\tan(2x+ct) = 4c^2\sec^2(2x+ct)\tan(2x+ct)$$

$0 = 0$ Satisfied.

Q No: 2: Expand the following function in a fourier series.

$$f(x) = x, \quad -\pi < x \le 0.$$

$$= 2x, \quad 0 \le x \le \pi.$$

Sol:-

Given function.

$$f x \begin{cases} x; & -\pi < x \le 0 \\ 2x; & 0 \le x \le \pi. \end{cases}$$

we have to find the fourier co-efficent, a_0 a_n & b_n.

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx.$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}.$$

$$\frac{1}{\pi}\left[0 - \frac{\pi^2}{2}\right] + \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$a_0 = \frac{-\pi}{2} + \pi = \frac{\pi}{2} \quad \text{--- (1)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(x\cos nx\,dx) + \frac{1}{\pi}\int_{0}^{\pi} 2x(\cos nx)\,dx.$$

$$\frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

So:

$$a_n = \begin{cases} \frac{-2}{\pi n^2} & ;\ \text{if } n \text{ is odd.} \\ 0 & ;\ \text{if } n \text{ is even.} \end{cases} \quad \text{— (2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx.$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[\frac{\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] = \frac{-3\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n} \rightarrow ③$$

So the required fourier series is:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

Q No:31- Solve the initial problem.

$$y'' - 4y' + 13y = 8\sin 3x, \quad y(0) = 1 \text{ and } y'(0) = 2.$$

Sol:-

$$y'' - 4y' + 13y = 8\sin 3x.$$

We have to find $y = y_c + y_p$.

for y_c the characteristic (auxiliary eqn) Eqn is:

$$m^2 - 4m + 13 = 0.$$

$$\Rightarrow m = \frac{4 \pm \sqrt{16-52}}{2} \Rightarrow m = \frac{4 \pm 6i}{2}$$

$$\Rightarrow m = 2 \pm 3i \quad ; \quad \alpha = 2 \,\&\, \beta = 3.$$

$$\text{So } y_c = e^{2x}\{c_1 \cos 3x + c_2 \sin 3x\}$$

for y_p Let

$$y_p = \text{Imag}\left(\frac{1}{m^2 - 4m + 13} 8e^{3ix}\right).$$

$$8 \text{ Imag } \frac{e^{3ix}}{-9 - 12i + 13}$$

$$8 \text{ Imag } \frac{e^{3ix}}{4 - 12i}$$

$$y_p = 2\,\text{Imag}\,\frac{e^{3ix}}{(1-3i)} \times \frac{(1+3i)}{(1+3i)}$$

$$y_p = 2\,\text{Imag}\left\{\frac{(1+3i)(e^{3ix})}{(1)^2-(3i)^2}\right.$$

$$y_p = 2\,\text{Imag}\,\frac{(1+3i)(e^{3ix})}{10}.$$

$$y_p = \frac{2}{10}\left(\sin 3x + 3\cos 3x\right)$$

So the general solution is

$$y = y_c + y_p.$$

$$y = c_1 e^{2x}\cos 3x + c_2 e^{2x}\sin 3x +$$

$$\frac{2}{10}\left(\sin 3x + 3\cos 3x\right).$$

Now use the initial condition $y(0) = 1$.

$$y(0) = c_1 e^{(0)}\cos(0) + c_2 e^{(0)}\sin(0) + \frac{2}{10}\left(\sin(0) + 3\cos(0)\right).$$

$$1 = c_1(1) + 0 + 0 + \frac{2}{10}\left(3(1)\right).$$

$$1 = c_1 + \frac{6}{10} \Rightarrow c_1 = 1 - \frac{6}{10} = \frac{4}{10} = 2/5.$$

Again use the another initial condition.

$$y'(0) = 2.$$

So.

$$y' = C_1 2e^{2x}\cos 3x + C_1 e^{2x}(-3\sin 3x) + C_2 2e^{2x}$$
$$\sin 3x + C_2 e^{2x}(3\cos 3x) + \frac{2}{10}(\cos 3x - 3\sin 3x)$$

$$y'(0) = C_1 2e^{(0)}\cos(0) + C e^{(0)}(-\sin(0) + C_2 2e^{(0)}\sin(0) +$$
$$C_2 e^{(0)}(3\cos(0) + \frac{2}{10}(\cos(0) - 3(\sin(0)).$$

$$2 = 2C_1 + 0 + 0 + C_2 3(1) + \frac{2}{10}(1 - 3(0)$$

$$2 = 2C_1 + 3C_2 + \frac{2}{10}$$

Use $C_1 = 2/5$.

$$2 = 2(2/5) + 3C_2 + \frac{2}{10}$$

$$\frac{1}{3}\left(2 - \frac{4}{5} - \frac{2}{10}\right) = C_2 \Rightarrow C_2 = \frac{1}{3}\left(\frac{20-8-2}{10}\right) = 1/3$$

So the General solution is.

$$y = 2/5\, e^{2x}\cos 3x + \frac{1}{3} e^{2x}\sin 3x + \frac{2}{10}\left[\sin 3x + 3\cos 3x\right]$$

b the required solution:

Q4: Solve:

$$(D^2 - DD')z = \cos x \cos 2y.$$

Sol:-

The auxiliary equation is:

$$m^2 - m = 0 \Rightarrow m = 0,\ m = 1.$$

Hence the complementry function is given by

$$Z_c = f_1(y) + f_2(y+x).$$

for the particular Integral, we have

$$Z_p = \frac{1}{D^2 - DD'} \cos x \cos 2y.$$

$$= \frac{1}{2} \frac{1}{D^2 - DD'} \left[\cos(x-2y) + \cos(x-2y)\right]$$

$$\frac{1}{2}\left[\frac{1}{D^2 - DD'} \cos(x+2y) + \frac{1}{D^2 DD'} \cos(x-2y)\right]$$

$$\frac{1}{2}\left[\frac{1}{-1+2} \cos(x+2y) + \frac{1}{-1-2} \cos(x-2y)\right]$$

$$\frac{1}{2} \cos(x+2y) - \frac{1}{6} \cos(x-2y)$$

Hence the complete solution is given by.

$$Z = f_1(Y) + f_2(Y+x) + \frac{1}{2} \cos(x+2y) - \frac{1}{6} \cos(x+2Y).$$

Ans.