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PAPER :- Differential Equation

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Question No. 1

(i) $w = \sin(x+ct) + \cos(2x+2ct)$

Sol:

Given $\dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2}$ —— ①

Now

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$= \frac{\partial}{\partial t}\left(\sin(x+ct)\right) + \frac{\partial}{\partial t}\left(\cos(2x+2ct)\right)$$

$$\frac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct)$$

Now

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[c\cos(x+ct) - 2c\sin(2x+2ct)\right]$$

$$\boxed{\frac{\partial^2 w}{\partial t^2} = -c^2\sin(x+ct) - 4c^2\cos(2x+2ct)}$$

Now

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - 2\sin(2x+2ct)$$

and

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct) - 2\sin(2x+2ct)\right]$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

① ⇒

$$-c^2\sin(x+ct) - 4c^2\cos(2x+2ct) =$$

$$c^2[-\sin(x+ct) - 4\cos(2x+2ct)]$$

$$-c^2\sin(\cancel{x+ct}) - 4c^2\cancel{\cos(2x+2ct)} = -c^2\sin(\cancel{x+ct}) - 4c^2\cancel{\cos(2x+2ct)}$$

$0 = 0$ (Satisfied)

(vi) $w = \tan(2x+ct)$

Now

$$\frac{\partial w}{\partial t} = c\sec^2(2x+ct)$$

and

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}(c\sec^2 2x+ct)$$

$$= c^2 2\sec^2(2x+ct)\tan(2x+ct)$$

Now

$$\frac{\partial w}{\partial x} = 2\sec^2(2x+ct)$$

$$\frac{\partial^2 w}{\partial x^2} = 4\sec^2(2x+ct)\tan(2x+ct)$$

① ⇒ $4c^2(\sec^2(\cancel{2x+ct}).\tan(2x+ct)) =$

$$4c^2\sec^2(\cancel{2x}+ct) + \tan(\cancel{2x}+ct)$$

$0 = 0$ Satisfied

Question No: 2

$f(x) = x; \; -\pi < x \leq 0$

$= 2x; \; 0 \leq x \leq \pi$

Sol: Given data

$$f(x) = \begin{cases} x; & -\pi < x \leq 0 \\ 2x; & 0 \leq x \leq \pi \end{cases}$$

we have to find the fourier coefficient a_0, a_n and b_n

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{\pi^2}{2}\right] + \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$\boxed{a_0 = -\frac{\pi}{2} + \pi = \frac{\pi}{2}} \quad (1)$$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[1-(-1)^n + 2(-1)^n - 2\right] = \frac{(-1)^n - 1}{\pi n^2}$$

So
$$\boxed{a_n = \begin{cases} -\frac{2}{\pi n^2}; & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even}\end{cases}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi} + \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$\boxed{b_n = \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] = \frac{-3\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}}$$

So the required Fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{\sin nx}{n}\right)$$

Q3 $y'' - 4y' + 13y = 8\sin 3x$.

Sol ~~we have~~ $y'' - 4y' + 13y = 8\sin 3x$

We have to ~~find~~ find $y = y_c + y_p$

for y_c: the characteristic (auxiliary eqn) eqns

$$m^2 - 4m + 13 = 0$$

$$m = \frac{4 \pm \sqrt{16-52}}{2} \Rightarrow m = \frac{4+6i}{2}$$

$\Rightarrow m = 2+3i \therefore \alpha = 2$ and $\beta = 3$

So $y_c = e^{2x}\{c_1 \cos 3x + c_2 \sin 3x\}$

For y_p let

$$y_p = \operatorname{Img}\left(\frac{1}{m^2-4m+13}\, 8e^{3ix}\right)$$

$$= 8 \operatorname{Img} \frac{e^{3ix}}{(3i)^2 - 4(3i) + 13}$$

$$= 8 \operatorname{Img} \frac{e^{3i}}{-9-12i+13}$$

$$= 8 \operatorname{Img} \frac{e^{3i}}{4-12i}$$

$$y_p = 2 \operatorname{Img} \frac{e^{3i}}{(1-3i)} \times \left(\frac{1+3i}{1+3i}\right)$$

$$y_p = 2 \operatorname{Img} \frac{(1+3i)(e^{3ix})}{(1)^2 - (3i)^2}$$

$$y_p = 2 \operatorname{Img} \frac{(1+3i)(e^{3ix})}{10}$$

$$y_p = 2/10\left(\operatorname{Img} (1+3i)(\cos 3x + i \sin 3x)\right)$$

$$y_p = \frac{2}{10}(\sin 3x + 3\cos 3x)$$

So the general solution is

$$y = y_c + y_p$$

$$y = C_1 e^{2x}\cos 3x + C_2 e^{2x}\sin 3x + \frac{2}{10}(\sin 3x + 3\cos 3x)$$

Now use the initial condition $y(0) = 1$

$$y(0) = C_1 e^{(0)}\cos(0) + C_2 e^{(0)}\sin(0) + \frac{2}{10}(\sin(0) + 3\cos(0))$$

$$1 = C_1(1) + 0 + 0 + \frac{2}{10}(3(1))$$

$$1 = C_1 + \frac{6}{10} \Rightarrow \boxed{C_1 = 1 - \frac{6}{10} = \frac{4}{10} = \frac{2}{5}}$$

Again use the another initial condition

$$y'(0) = 2$$

so $y' = C_1 2e^{2x}\cos 3x + C_1 e^{2x}(-3\sin 3x)$

$+ C_2 2e^{2x}\sin 3x + C_2 e^{2x}(3\cos 3x)$

$+ \frac{2}{10}(\cos 3x - 3\sin 3x)$

$$y'(0) = C_1 2e^{0}\cos(0) + C_1 e^{(0)}(-3\sin(0))$$

$$+ C_2 2e^{(0)}\sin(0) + C_2 e^{(0)}(3\cos(0))$$

$$+ \frac{2}{10}\left(\cos(0) - 3\sin(0)\right)$$

$$2 = 2C_1 + 0 + 0 + C_2 3(1) + \frac{2}{10}\left(1 - 3(0)\right)$$

$$2 = 2C_1 + 3C_2 + 2/10$$

$$2 = 2(2/5) + 3C_2 + 2/10$$

$$2 = 2(2/5) + 3C_2 + 2/10 \qquad \boxed{\text{use } C_1 = 2/5}$$

$$\frac{1}{3}\left(2 - 4/5 - 2/10\right) = C_2 \Rightarrow \boxed{C_2 = 1/3\left(\frac{20-8-2}{10}\right) = 1/3}$$

So the general solution is

$$y = 2/5\, e^{2x}\cos 3x + 1/3\, e^{2x}\sin 3x + 2/10\left[\sin 3x + 3\cos 3x\right]$$

is the required solution

QNO:4

$$(D^2 - DD')z = \cos x \cos 2y$$

Sol..
the auxiliary equation is

~~m2-~~

$$m^2 - m = 0 \Rightarrow m = 0$$

$$m = 1$$

Hence the complementary function is given by

$$z_c = f_1(y) + f_2(y+x)$$

For the particular integral we have

$$z_p = \frac{1}{D^2 - DD'}\left[\cos(x-2y) + \cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\cos(x+2y) + \frac{1}{D^2 - DD'}\cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{-1+2}\cos(x+2y) + \frac{1}{-1-2}\cos(x-2y)\right]$$

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$$= \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Hence the complete solution is given by

$$z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$