Final Paper

Submitted By:
Yahya Riaz (12280)
BSSE Section A

Submitted To:
Sir Mansoor Qadir

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Paper:
Linear Algebra

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Summer Final Exam **Total: 50 Marks**

Note: Submission time 25-09-2020 before 6:00 pm (3 Hrs)

Students who have not attempted mid term exam, must download and solve 80 marks paper only

Question No: 1 **10 marks**

Find the eigenvalues of A

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{4} & -\mathbf{17} & \mathbf{8} \end{bmatrix}$$

Question No. 2 **10 marks**

Find a matrix P that diagonalizes the below matrix

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{3} \end{bmatrix}$$

Question No. 3 **10 marks**

Determine whether the vectors form linear dependent or independent sets.

V1 = (1, -2, 3)

V2 = (5, 6, -1)

V3 = (3, 2, 1)

Question No. 4 **20 marks**

What are the four main things we need to define for a vector space? Which of the following is a vector space over R? For those that are not vector spaces, modify one part of the definition to make it into a vector space.

a. V = { 2 x 2 matrices with entries in R}, usual matrix addition, and

$$k.\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & b \\ kc & d \end{pmatrix} for\ k \in R$$

b. V = {Polynomials with complex coefficients of degrees ≤3}, with usual addition and scalar multiplication of polynomials.

Question #1:
Answer:

Question # 01:-

let $[A - \lambda I] = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -17 & 8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 1 \\ -17 & 8-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 4 & 8-\lambda \end{vmatrix} = 0$$

$$-\lambda\left[-\lambda(8-\lambda) + 17\right] - 1(-4) = 0$$

$$-\lambda\left[-8\lambda + \lambda^2 + 17\right] + 4 = 0$$

$$-\lambda^3 + 8\lambda^2 - 17\lambda + 4 = 0$$

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

$\lambda = 4$ is a root

$$\begin{array}{c|cccc}
4 & 1 & -8 & 17 & -4 \\
 & \downarrow & 4 & -16 & 4 \\
\hline
 & 1 & -4 & 1 & \boxed{0}
\end{array}$$

$$\lambda^2 - 4\lambda + 1 = 0$$

$$\lambda = 4 \pm \sqrt{\frac{16 - 4(1)(1)}{2}}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$\boxed{\lambda = 2 \pm \sqrt{3}}$$

Question #2:
Answer:

Question # 02

Solution :-

A can be diagonalized if there exists an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$

Here $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

finding eigenvalues of the matrix A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} (-\lambda) & 0 & -2 \\ 1 & (2-\lambda) & 1 \\ 1 & 0 & (3-\lambda) \end{vmatrix} = 0$$

$$(-\lambda)\Big((2-\lambda)(3-\lambda) - 1(0)\Big) - 0\Big(1(3-\lambda) - 1(1)\Big)$$

$$+ \Big((-2)\; 1(0) - (2-\lambda)(1)\Big) = 0$$

$$-\lambda\big((6 - 5\lambda + \lambda^2) - 0\big) - 0\big((3-\lambda) - 1\big) - 2\big(0 - (2-\lambda)$$

$-\lambda\left((6-5\lambda+\lambda^2) - 0(2-\lambda) - 2(-2+\lambda)\right) = 0$

$(-6\lambda+5\lambda^2-\lambda^3) - 0 - (-4+2\lambda) = 0$

$(-\lambda^3+5\lambda^2-8\lambda+4) = 0$

$-(\lambda-1)(\lambda-2)(\lambda-2) = 0$

So

$\lambda-1=0, \quad \lambda-2=0, \quad \lambda-2=0$

The eigenvalues of the matrix A are given by $\lambda = 1, 2$.

$\therefore$ 2 Eigenvectors of $\lambda = 2$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The eigenvectors compose the columns of matrix. P

$$P = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The diagonal matrix D is composed of eigenvalues.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\rightarrow$ Now find P^{-1}

$$|P| = \begin{vmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= -2(1\times1-0\times0)+0(1\times1-0\times1)-1(1\times0-1\times1)$$

$$= -2(1+0)+0(1+0)-1(0-1)$$

$$= -2(1)+0(1)-1(-1)$$

$$= -1$$

Question #3:
Answer:

Question # 03:-

Solution:-

Here $A = (1, -2, 3), B = (5, 6, 1)$

$C = (3, 2, 1)$

The vectors A, B, C are linearly dependent if their determinant is Zero $|D| = 0$.

$$|D| = \begin{vmatrix} 1 & -2 & 3 \\ 5 & 6 & -1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 6 & -1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 5 & -1 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 5 & 6 \\ 3 & 2 \end{vmatrix}$$

$$= 1(6 \times 1 - (-1) \times 2) + 2(5 \times 1 - (-1) \times 3) + 3(5 + 2 - 6 \times 3)$$

$$= 1(8+2)(8) + 3(-8)$$

$$= 8 + 16 - 24$$

$$= 0$$

→ since $|D| = 0$, so vectors A, B, C are linearly independent.

Question #4 (a):

A vector is a set V on which two operation + and - are defined, called vector addition and scalar multiplication.

The operation + (vector addition) must satisfy the following conditions:

Closure: if u and v are any vectors in V, then the sum u + v belongs to V.

1. *Commutative law*: For all vectors u and v in V, u + v=v + u.
2. *Associative law*: For all vectors u, v, w in V, u + (v + w) = (u + v) + w.
3. *Additive identity*: The set V contains an additive identity element, denote by 0, such that for any vector v in V, 0 + v=v and v + 0=v.
4. *Additive inverses*: for each vector v in V, the equations v + x = 0 and x + v = 0 have a solution x in V, called an additive inverse of v, and denoted by – v.

The set Pn(x) of all the polynomials over R in variable x of degree <= n forms a vector space over R.

If $F_n(x)$ = a0 + a1x +…… + anxn

And $g_n(x) = b_0 + b_1x +\ldots\ldots. +b_n\ x_n,\ a_i,\ b_i$ E R

Then $F_n(x) + g_n\ (x) = (a_0 + b_0) + (a_1 + b_1)\ x +\ldots\ldots\ (a_n + b_n)\ x^n$

The associative additive property is included from the additive associative property of R.

The zero polynomial f(x) = 0 of degree 0 acts as the additive identity of P(x) and $–f(x) = -a_0 + (-a_1)\ x +\ldots. + (-a_n)\ x^n$ additive inverse of $f_n(x)$.

Commutative property follows from the commutative property of R. Hence $P_n(x)$ is an additive abelian group.

The scalar multiplication of a E R by $f_n\ (x) = aa_0 + (aa_1)\ x + (aa_2)\ x_2\ldots. + (aa_n)\ x^n$ E $P_n(x)$.

it observes properties of scalar multiplication which can easily be verified. So that $p_n(x)$ forms a vector space over R.

Question #4 (b):

Question #4(b)

let F be a field, A non-empty set V together with two binary operations (+) and (·) form the algebraic structure of a vector space over the field F, if

(a) V forms an additive (+) abelian group

(b) The scalar multiplication (·) as a function from $F \times V$ into V observes the following properties:

(i) $\forall\, \alpha \in F,\ x, y \in V,\ \alpha(x+y) = \alpha x + \alpha y$

(ii) $\forall\, \alpha, \beta \in F,\ (\alpha+\beta)x = \alpha x + \beta x \quad \forall x \in V$

(iii) $\forall\, \alpha, \beta \in F,\ \forall\, x \in V;\ \alpha(\beta x) = (\alpha$

(iv) For e_F, the identity of F, $e_F x = x$

$\forall x \in V$.