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Section = "B"

Sub = diff equation

Assignment = 03

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(1)

Application of Partial differential equation:

Many engineering problems are governed by different types of partial differential equations, and some of the more important types are given below.

Tricomi equation:-

$$y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \begin{cases} y > 0 : \text{elliptic} \\ y < 0 : \text{hyperbolic} \end{cases}$$

Laplace equation (or variants): $\partial^2 \phi / \partial x^2 + \partial^2 \phi / \partial y^2 = \nabla^2 \phi = 0$

Poisson's equation:-

$$\partial^2 \phi / \partial x^2 + \partial^2 \phi / \partial y^2 = f(x, y)$$

Helmholtz equation:

$$\partial^2 \phi / \partial x^2 + \partial^2 \phi / \partial y^2 + C^2 \phi = 0$$

Plate bending:-

$$\nabla^2 \nabla^2 w = \nabla^4 w = q/D$$

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Wave equation:-

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

Fourier equation:-

$$\partial T \partial t = a (\partial^2 T \partial x^2)$$

Separable differential equations:-

For equations which can be expressed in separable form as shown below, the solution can be obtained easily as,

$$\frac{dy}{dx} = F(x, y) \quad \frac{dy}{g(y)} = f(x) dx \quad \int \frac{dy}{g(y)} =$$

$$\int f(x) dx + C$$

$$M(x, y) dx + N(x, y) dy = 0 \quad M(x) dx = N(y) dy$$

$$\text{then } \int M(x) dx = - \int N(y) dy + C.$$

3)

Separable differential equations:-

For equations which can be expressed in separable form as shown below, the solution can be obtained easily as.

$$\frac{dy}{dx} = F(x, y) \quad \frac{dy}{g(y)} = f(x) dx \quad \int \frac{dy}{g(y)} = \int f(x) dx + C$$

$$(x) dx + C$$

$$M(x, y) dx + N(x, y) dy = 0 \quad M(x) dx = -N(y) dy$$

$$\text{then } \int M(x) dx = - \int N(y) dy + C$$

Example:-

$$\frac{dy}{dx} = 8x^3 + (y^2 + 1) \Rightarrow \frac{dy}{y^2 + 1} = 8x^3 dx$$

$$\int \frac{dy}{y^2 + 1} = \int 8x^3 dx + C \Rightarrow \tan^{-1} y = \frac{1}{4} x^4 + C$$

$$\Rightarrow y = \tan \left(\frac{1}{4} x^4 + C \right)$$

Example:-

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \quad \text{subject to } y(0) = -1$$

4)
Since is a separable function, the problem can be solved as.

$$2(y-1) dy = (3x^2 + 4x + 2) dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

Example:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \quad \text{subject to } y(0) = -1$$

Since this is a separable function, the problem can be solved as

$$2(y-1) dy = (3x^2 + 4x + 2) dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

Based on the boundary condition, $c = 3$,

$$\text{hence } y^2 - 2y = x^3 + 2x^2 + 2x + 3.$$

This quadratic equation in y^2 can be solved with two solutions by the quadratic equation as:

5)

$$y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4} \quad \text{and} \quad y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4}$$

Since the second solution does not satisfy the boundary condition, it will not be accepted; hence, the solution to this differential equation is obtained.

Variation of parameters:-

for the following equation form, it is possible to solve it by variation of parameters.

$$\text{for } \frac{dy}{dx} = P(x)y + Q(x)$$

Let $y = C(x)e^{\int P(x) dx}$ by differentiating,

it gives.

$$\frac{dy}{dx} = \frac{dC(x)}{dx} e^{\int P(x) dx} + \underbrace{C(x)P(x)e^{\int P(x) dx}}_{P(x)y}$$

Substitute it to the original ODE

6)

$$\frac{dc(x)}{dx} = Q(x) e^{-\int P(x) dx} \quad \text{Comparing the}$$

terms, it given

$$C(x) = \int Q(x) e^{-\int P(x) dx} dx + \bar{C}$$

Example:-

$$(x+1) \frac{dy}{dx} - ny = e^x (x+1)^{n+1}$$

This equation is now expressed as,

$$\frac{dy}{dx} = P(x) y + Q(x)$$

$$\frac{dy}{dx} = \frac{n}{x+1} y + \frac{e^x (x+1)^{n+1}}{x+1}$$

for $x \neq -1$

Solving the homogeneous part of the ODE

$$\frac{dy}{dx} = \frac{n}{x+1} y \quad \text{then} \quad \frac{dy}{y} = \frac{n}{x+1} dx$$

$$\ln |y| = \ln |x+1|^n + c_1$$

$$y = e (x+1)^n$$

7)

Look for solution $y = c(x) (x+1)^n$

where $c(x)$ is the variation of parameters.

Substitute it to the ODE.

$$\frac{dc(x)}{dx} (x+1)^n + nc(x) (x+1)^{n-1} = nc(x)$$

$$(x+1)^{n-1} = nc(x) (x+1)^{n-1} + e^x (x+1)^n$$

$$\frac{dy}{dx} = \frac{n}{x+1} y + e^x (x+1)^n$$

Comparison gives $\frac{dc(x)}{dx} = e^x$

Integration of this equation gives.

$$c(x) = e^x + \bar{C}$$

General solution is hence given by

$$y = (x+1)^n (e^x + \bar{C})$$

8) The Bernoulli equation is an important equation type which can be solved in a similar way by variation of parameters. Consider the following form of equation.

$$\frac{dy}{dx} = P(x)y + Q(x)y^n$$

Step 1:- put $z = y^{1-n}$

Step 2:- Then $\frac{dz}{dx} = (1-n)y^{-n} \frac{dx}{dy}$

$$\frac{dz}{dx} = (1-n)P(x)z + (1-n)Q(x)$$

The non linear ODE now becomes linear ODE. It can be solved by formula.

Step 3:- $n = -1, z = y^2$. Inverting z to get y

$$\frac{dy}{dx} = \frac{y}{2x} + \frac{x^2}{2y}$$

$$\frac{dz}{dx} = \frac{1}{x}z + x^2$$

9)

$$z = e^{\int \frac{1}{x} dx} \left(\int x^2 e^{-\int \frac{1}{x} dx} dx + c \right) = Cx + \frac{1}{2} x^2$$

Back substitution of $z = y^2$ of $z = y^2$

$$y^2 = Cx + \frac{1}{2} x^2$$

Homogeneous equations:-

for equation of the following type, where all the coefficients are constant, it can be evaluated according to different.

Laplace equation.

Laplace equation forms an important governing condition for many types of problems. Some of the more common forms are given by.

Three dimensional Laplace equation:-

$$u_{xx} + u_{yy} + u_{zz} = 0$$

10)

Two dimensional heat Conduction:-

$$\alpha^2 (u_{xx} + u_{yy}) = u_t$$

Two dimensional Seepage Problem.

$$(K_x u_{xx} + K_y u_{yy}) = 0$$

There are two major types of boundary conditions to this problem.

Dirichlet problem:-

Boundary conditions prescribed

as u .