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Questions # 1

$x^3 y''' + 2x^2 y' + 2y - 10x + \frac{10}{x}$

Solution

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{dy}{dx} + 2y = 10x + 10x^{-1}$$

$$x^3 D^3 y + 2x^2 D^2 + 2y = 10x + 10x^{-1}$$

$$(x^3 D^3 + 2x^2 D^2 + 2) y = 10x + 10^{-1} x$$

$$(x^3 D^3 + 2x^2 D + 2) y = 10x + 10^{-1} x \quad \text{— } 1$$

let $x = e^t \Rightarrow t = \ln x$

$$xD = \Delta$$

$$x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$$

$$x^3 D^3 = \Delta(\Delta - 1)(\Delta - 2)$$

Subtituting into eq — 1

$$(\Delta - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta) + 2) y =$$

$$\cancel{10^x + 10}\ 10x - 10x^{-1}$$

$$(\Delta^3 - \Delta^2 + 2)y = 10x + 10x^{-1}$$

$$(m^3 - m^2 + 2)y = 10e^t + \frac{10}{et}$$

Using synthetic division

$$\begin{array}{r|rrr|r} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

$$\Delta^2 - 2\Delta + 2 = 0$$

Now using Quadratic Formula

$a = 1$, $b = 2$, $c = 2$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta = \frac{-(2) \pm \sqrt{-2^2 - 4(1)(2)}}{2a}$$

(3)

$$\Delta = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\Delta = \frac{2 \pm \sqrt{-4}}{2}$$

$$\Delta = \frac{2 \pm \sqrt{-4}}{2}$$

$$\Delta = \frac{2 \pm \sqrt{-1} + \sqrt{4}}{2}$$

$$\Delta = \frac{2 + 2i}{2}$$

$$\Delta = \cancel{2}\frac{(1 \pm i)}{\cancel{2}}$$

$$\Delta = 1 \pm i$$

Since roots are complex

$$y_c = e^{-x}(c_1 \cos t + c_2 \sin t)$$

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Now particular integration

$$y_P = \frac{1}{\Delta^3-\Delta^2+2} 10e^t + \frac{1}{\Delta^3-\Delta^2+2} 10/e^t$$

$$= \frac{10e^t}{(1)^3-(1)^2+2} + \frac{10e^{-t}}{(1)^3-(1)^2+2}$$

$$= \frac{\overset{5}{\cancel{10}}e^t}{\cancel{2}} + \frac{\overset{5}{\cancel{10}}e^{-t}}{\cancel{2}}$$

$$= 5e^t + 5e^t$$

$$y_P = 5e^t + 5e^t$$

General solution

$$y = y_c + y_P$$

$$y = e^{-x}(c_1 \cos t + c_2 \sin t) + 5e^t + 5e^t$$

put $e^t = x$ and $t = \ln x$

$$y = e^{-x}(c_1 \tan x + c_2 \sin/x) + 5e^x + 5e^{-x}$$

Question # 2

$$x^3 \frac{d^3y}{dx} + 4x^2 \frac{d^2y}{dx} - 5x\frac{dy}{dx} - 15y = x^4$$

Solution

let $\frac{d}{dx} = \Delta$

$$x^3\Delta^3 y + 4x^2\Delta^2 y - 5x\Delta y - 15y = x^4$$

$$(x^3\Delta^3 + 4x^2\Delta^2 - 5x\Delta - 15)y = x^4$$

let

$$x = e^t \Rightarrow t = \ln x$$

$$x\Delta = \Delta$$

$$x^2\Delta^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$$

$$x^3\Delta^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$$

Now subtituting

$$(x^3\Delta^3 + 4x^2\Delta^2 - 5x\Delta - 15)y = x^4$$

$$(\Delta^3 - 3\Delta^2 + 2\Delta + 4(\Delta^2 - \Delta) - 5(\Delta) - 15)y = e^{4t}$$

$$(\Delta^3 + \Delta^2 + 7\Delta - 5)y = e^{4t}$$

Synthetic division

	1	+1	-7	-15
5		3	12	15
	1	4	5	0

$$\Delta^2 + 4\Delta + 5 = 0$$

Quadratic Formula

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$\Delta = \frac{\cancel{2}(-2 \pm 1)}{\cancel{2}}$$

7

$y_c = e^{4t}\,(c_1 \cos t + c_2 \sin t)$

for $y_p = ?$

$$y_p = \frac{1}{\Delta^3 + \Delta^2 - 7\Delta - 15}\, e^{4t}$$

$$= \frac{1}{(4)^3 + (4^2) - 7(4) - 5}\, e^{4t}$$

$$= \frac{1}{64 + 16 - 28 - 15}\, e^{4t}$$

$$= \frac{1}{80 - 43}\, e^{4t}$$

$$y_p = \frac{1}{37}\, e^{4t}$$

Hence

$$y = y_c + y_p$$

$$y = (c_1 \cos t + c_2 \sin t) + \frac{1}{37}\, e^{4t}$$

again put $t = \ln x$ & $x = \ln x$

$$y = e^{3t}\,(c_1 \cos t \ln x + (c_2 \sin \log x) + \frac{1}{37} e^{4x}$$

Question # 3

$x^2 y''' + 2xy' - by = 10x^2$

Solution

$y(1) = 1$ and $y'(1) = -6$

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - by = 10x^2$$

$$\Rightarrow \left(x^2 d^2/dx^2 + 2x d/dx - 6\right) y = 10x^2$$

Put $x\Delta = \Delta \Rightarrow x^2 \Delta^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$x = e^t$ and $\log x = t$

$(\Delta^2 - \Delta + 2\Delta - 6) y = 10 e^{2t}$

$(\Delta^2 + \Delta - 6) y = 10 e^{2t}$

The characteristic equation

$\Delta^2 + \Delta - 6 = 0$

$\Delta^2 + 3\Delta - 2\Delta - 6 = 0$

$\Rightarrow \Delta(\Delta+3) - 2(\Delta+3) = 0$

$\Rightarrow (\Delta+3)(\Delta-2) = 0$

$\Delta+3=0 \quad , \Delta-2=0$

$\Delta = 2 \qquad \Delta = -3$

Since roots are real & distinct

For $y_c = ?$

$y_c = C_1 e^{-3t} + C_2 e^{2t}$

For $y_p = ?$

$$y_p = \frac{1}{\Delta^2 - \Delta - 6} 10^{2t}$$

$$= \frac{10}{\Delta^2 - \Delta - 6} e^{2t}$$

$$= 10 \frac{1}{0} e^{2t} \text{ fails}$$

Now

$$10 \frac{1}{d/d\Delta(\Delta^2 + \Delta - 6)} e^{2t}$$

$\Rightarrow 10 \frac{t}{2D+1} e^{2t}$

$= 10 \frac{1.t}{4+1} e^{2t}$

$y_P = 2te^{2t}$

General Solution

$y = y_c + y_P$

$= c_1 e^{-3t} + c_2 e^{2t} + 2te^{2t}$

$y = c_1 x^{-3} + c_2 x^2 + 2(\log x) x^2$ —— (B)

Put $y(1) = 1$ i.e $x = 1, y = 1$ in —— (B)

$1 = c_1 (1)^{-3} + (c_2 (1)^2 + 2 \log(1))$

$1 = c_1 + c_2$ —— (C)

Now differentiation eq —— (B)

w.r.t x

$y' = -3c_1 x^{-4} + 2\left(2x + \frac{2}{x}(x)^2 + 4x\log x\right.$

Now put $y'(1) = -6$ i-e $y' = -6$ and $x = -6$

$-6 = -3c_1 + 2c_2 + 2 + 0$

$\Rightarrow -6 = -3c_1 + 2c_2 + 2$

$\Rightarrow -6 - 2 = -3c_1 + 2c_2 + 2$

$-8 = 3c_1 + 2c_2$ — (1)

xing eq (1) with (2) & ÷ing from (1)

$2c_1 + \cancel{2c_2} = 2$

$\mp 3c_1 \mp \cancel{2c_2} = \mp 8$

$5c = 10$

$c_1 = \frac{10}{5}$ $c_1 = 2$

$-8 = -3(2) + 2c_2$

$-8 = -6 + 2c_2$

$2c_2 = -8 + 6$

$2c_2 = -2$

$$c_2 = -\frac{\cancel{2}}{\cancel{2}} 1$$

$$c_2 = -1$$

Now put the value of c_1

& c_2 in eq_B

$$y = 2x^{-3} - x^2 + 2\ln x \; x(x^2)$$

$$y = \frac{2}{x^3} - x^2 + 2x^2 \log x$$

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Question # 4

$x^2 y'' + 7xy' + 5y = x^5$

$y(0) = 2$ and $y(1) = 2$

Solution

$$x^2 \frac{dy^2}{dx^2} + \frac{7x\,dy}{dx} + 5y = x^5$$

$$\Rightarrow \left(x^2 \frac{d^2}{dx^2} + 7\frac{x\,d}{dx} + 5\right)y = x^5 \quad \text{— (A)}$$

Put $xD = \Delta \Rightarrow x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

$x = e^t \Rightarrow \log x = t$ in eq — (A)

$\Rightarrow (\Delta^2 - \Delta + 7\Delta + 5)y = e^{5t}$

$\Rightarrow (\Delta^2 + 6\Delta + 5)y = e^{5t}$

By Quadratic Formula

$$\Delta = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36-20}}{2}$$

$$= \frac{-6 \pm \sqrt{16}}{2}$$

$$= \frac{-6 \pm \sqrt{4^2}}{2}$$

$$= \frac{2(-3 \pm 2)}{2}$$

$D = -3 \pm 2$ since roots are real and distinct

$\cancel{y_c = -3 \pm 2}$

$y_c = C_1 e^{-5t} + C_2 e^{-t}$

for $y_p = ?$

$$y_p = \frac{1}{D^2+6D+5} e^{5t}$$

$$= \frac{1}{(5)^2+6(5)+5} e^{5t}$$

$$= \frac{1}{60} e^{5t}$$

Now general solution is

$$y = y_c + y_p$$

$$y = c_1 e^{-5t} + c_2 e^{-t} + \frac{1}{60} e^{5t}$$

$$y = c_1 x^{-5} + c_2 x^{-1} + \frac{1}{60} x^{5} \quad \text{— (B)}$$

$x=0$ put in this equation

No in eq (B) $\quad e^0 = 1$

Put $y(0) = 2$ i-e $y = 2$ and $x = 2$

$$2 = c_1(2)^{-5} + 2(2)^{-1} + \frac{1}{60}(2)^5$$

(16)

$$2 = -32c_1 - 2c_2 + \frac{1}{\cancel{60}\,15}(3\overset{8}{\cancel{2}})$$

$$2 = -32c_1 - 2c_2 + \frac{8}{15}$$

$$2 - \frac{8}{15} = -32c_1 - 2c_2$$

$$\frac{22}{15} = -32c_1 - 2c_2 \quad \text{---} \; ⓒ$$

Now differentiate eq Ⓑ w.r.t (x)

$$y' = -5c_1 x^{-6} - c_2 x^{-2} + \frac{1}{12}x^4 \text{---}*$$

Put $y'(1) = 2$ i-e $y' = 2$ and $x = 2$ in above equation

$$2 = -5c_1 \, c_2^{-6} - c_2 (2)^{-2} + \frac{1}{12}(2)^4$$

$$2 = -5c_1(-64) - c_2(4) + \frac{1}{12}(16)$$

$$2 = 320c_1 + 4c_2 + 4/3$$

$\Rightarrow 2 - \frac{4}{3} = 320c_1 + 4c_2$

$\Rightarrow 2/3 = 320c_1 + 4c_2$ — (D)

xing eq (C) with 2 and then ÷ing

eq (C) from (D)

$\frac{-44}{15} = 64c_1 + 4c_2$

$\frac{-44}{15} = 64c_1 + 4c_1$

$\underline{\mp 2/3 = \pm 320c_1 \pm 4c_2}$

$\frac{34}{5} = -256c_1$

$c_1 = \frac{34}{15} \times 256$

$c_1 = 580$

Put the value of c_1 in eq (C)

$\frac{22}{15} = -\frac{32}{} (580) - 2c_2$

$$\Rightarrow \frac{22}{15} = -18560 - 2C_2$$

$$\Rightarrow \frac{22}{15} = +18560 = -2C_2$$

$$\Rightarrow \frac{18560}{-2} = C_2$$

$$\boxed{-9280 = C_2}$$

Now put the value of C_1 & C_2 in eq (B)

$$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x^5$$

$$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5$$

Question # 5

$(x+1)^2 y'' - 3(x+1)' y' + 4y = x^2$

Solution

$$(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2$$

$$\Rightarrow \left((x+1)^2 \frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4\right)y = x^2$$

$$\Rightarrow \left((x+1)^2 D^2 - 3(x+1)D + 4\right)y = x^2 \quad \text{—— (A)}$$

Put $(x+1)D = \Delta \Rightarrow (x+1)^2 D^2 = \Delta(\Delta-1)$.

$x = e^t$ in eq (A)

$$\Rightarrow (\Delta^2 - 4\Delta + 4)y = e^{2t}$$

For y_c we find the roots

$\Delta^2 - 4\Delta + 4 = 0$

$\Delta^2 - 2\Delta - 2\Delta + 4 = 0$

$\Delta(\Delta-2) - 2(\Delta-2) = 0$

$D-2=0$, $D=2$

$D-2=0$, $D=2$

So the roots are real and reapet

The General Solution are

$y = (c_1 + c_2 x)^{2x}$

$y = (c_1 + c_3 x)^{2x}$

For $y_P = ?$

$$y_P = \frac{1}{D^2 - 4D + 4} \quad \Big| \quad (2)^2 - 4(2) + 4 \Rightarrow 0$$

$$y_P = \frac{2}{2D-4} e^{2t}$$

(If we put 2

$2D - 4 \Rightarrow 2(2) - 4 = 0$

we take again derivation

$y_P = \frac{\cancel{2}}{\cancel{2}} e^{2t}$

$y = (c_1 + c_2 x)^{2t} + e^{2t}$ → General solution.