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Section:

"A"

Subject:

Differential Equations

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Answer No 1

(i) $w = \sin(x+ct) + \cos(2x+2ct)$

Solution:

As given

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \quad \text{--- (1)}$$

Now

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$= \frac{\partial}{\partial t}(\sin(x+ct)) + \frac{\partial}{\partial t}(\cos(2x+2ct))$$

$$\frac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct)$$

Now

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[c\cos(x+ct) - 2c\sin(2x+2c\right.$$

$$\frac{\partial^2 w}{\partial t^2} = -c^2\sin(x+ct) - 4c^2\cos(2x+2ct)$$

Now

$$\frac{dw}{dx} = \frac{d}{dx}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$\frac{dw}{dx} = \cos(x+ct) - 2\sin(2x+2ct)$$

$$\frac{d^2w}{dx^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$1 \Rightarrow -c^2\sin(x+ct) - 4c^2\cos(2x+2ct)$$

$$= c^2\left[-\sin(x+ct) - 4\cos(2x+2ct)\right]$$

$$-c^2\cancel{\sin}(x+ct) - 4c^2\cancel{\cos(2x+2ct)} = -c\,\cancel{\sin}(x+ct)$$
$$-4c^2\cos(\cancel{2x}+2ct)$$

$$c = 0 \quad (\text{Satisfied})$$

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Answer No 1 (ii)

$w = \tan(2x + ct)$

Solution:

$w = \tan(2x+ct)$

Now

$$\frac{\partial w}{\partial t} = c\sec^2(2x+ct)$$

and $$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(c\sec^2(2x+ct)\right)$$

$$= c^2 \cdot 2\sec^2(2x+ct)\tan(2x+ct)$$

Now

$$\frac{\partial w}{\partial x} = 2\sec^2(2x+ct)$$

$$\frac{\partial^2 w}{\partial x^2} = 4\sec^2(2x+ct)\tan(2x+ct)$$

$$\cancel{\frac{\partial^2 w}{\partial x^2} = }$$

(1) $\Rightarrow$ $4c^2\sec^2(2x+ct)\tan(2x+ct)$

$= 4c^2\sec^2(2x+ct)\tan(2x+ct)$

$0 = 0$ (Satisfied)

Answer No 2

Solution:

Given function is

$$F(x) = \begin{cases} x \; ; & -\pi < x \le 0 \\ 2x \; ; & 0 \le x \le \pi \end{cases}$$

We have to find the fourier co-efficients, a_0, a_n and b_n

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$a_0 = \frac{1}{\pi}\left[0 - \frac{\pi^2}{2}\right] + \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$\boxed{a_0 = \frac{-\pi}{2} + \pi = \frac{\pi}{2}} \longrightarrow 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(x\cos nx\,dx) + \frac{1}{\pi}\int_{0}^{\pi}(2x\cos nx)\,dx$$

$$a_n = \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+\frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

So

$$\boxed{a_n = \begin{cases} \frac{-2}{\pi n^2} ; & \text{if } n \text{ is odd} \\ 0 ; & \text{if } n \text{ is even} \end{cases}} \rightarrow (2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$$

$$+\frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$\boxed{\begin{aligned} b_n &= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] \\ &= \frac{-3\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n} \end{aligned}} \longrightarrow (3)$$

The required Fourier Series is

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

Answer No 3

Solution:

$$y'' - 4y' + 13y = 8\sin 3x$$

We have to find $y = y_c + y_p$

For y_c the characteristic eq is

$$m^2 - 4m + 13 = 0$$

$$\Rightarrow m = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$\Rightarrow m = \frac{4 + 6i}{2}$$

$$\Rightarrow m = 2 \pm 3i \; , \; \alpha = 2, \text{ and } \beta = 3$$

$$\Rightarrow y_c = e^{2x}[c_1 \cos 3x + c_2 \sin 3x)$$

For y_p Let

$$y_p = \mathrm{Imag}\left(\frac{1}{m^2 - 4m + 13} 8e^{3ix}\right)$$

$$y_p = 8\,I_{mag}\, \frac{e^{3ix}}{(3i)^2 - 4(3i) + 13}$$

$$= 8\,I_{mag}\, \frac{e^{3ix}}{-9 - 12i + 13}$$

$$= 8\,I_{mag}\, \frac{e^{3ix}}{4 - 12i}$$

$$y_p = 2\,I_{mag}\, \frac{e^{3ix}}{(1-3i)} \times \frac{(1+3i)}{(1+3i)}$$

$$y_p = 2\,I_{mag}\, \frac{(1+3i)(e^{3ix})}{(1)^2 - (3i)^2}$$

$$y_p = 2\,I_{mag}\, \frac{(1+3i)(e^{3ix})}{10}$$

$$y_p = \frac{2}{10}\left(I_{mag}(1+3i)(\cos 3x + \sin 3x)\right)$$

$$y_p = \frac{2}{10}\left(\sin 3x + 3\cos 3x\right)$$

The general solution is

$$y = y_c + y_p$$

$$y = C_1 e^{2x}\cos 3x + C_2 e^{2x}\sin 3x + \frac{2}{10}\left[\sin 3x + 3\cos 3x\right]$$

Now use the initial condition
$y(0) = 1$

$y(0) = C_1 e^{(0)} \cos(0) + C_2 e^{(0)} \sin(0) + \frac{2}{10} [\sin(0)$
$+ 3\cos(0)]$

$1 = C_1(1) + 0 + 0 + \frac{2}{10}(3(1))$

$1 = C_1 + \frac{6}{10}$

$C_1 = 1 - \frac{6}{10} = \frac{10-6}{10} = \frac{4}{10} = \frac{2}{5}$

Now use the another initial condition
$y'(0) = 2$

$y' = C_1 2e^{2x}\cos 3x + C_1 e^{2x}(-3\sin 3x)$
$+ C_2 2e^{2x}\sin 3x + C_2 e^{2x}(3\cos 3x) + \frac{2}{10}$
$(\cos 3x - 3\sin 3x)$

$y'(0) = C_1 2e^{(0)}\cos(0) + C_1 e^{(0)}(-3\sin(0) + C_2$
$2e^{(0)}\sin(0) + C_2 e^{(0)}(3\cos(0)) + \frac{2}{10}$
$(\cos(0) - 3\sin(0))$

$$2 = 2C_1 + 0 + 0 + C_2 3(1) + \frac{2}{10}(1-3(0))$$

$$2 = 2C_1 + 3C_2 + \frac{2}{10}$$

$$2 = 2\left(\frac{2}{5}\right) + 3C_2 + \frac{2}{10}$$

$$\frac{1}{3}\left(2 - \frac{4}{5} - \frac{2}{10}\right) = C_2$$

$$\Rightarrow C_2 = \frac{1}{3}\left(\frac{20-8-2}{10}\right) = \frac{1}{3}$$

So the general equation is

$$y = \frac{2}{5}e^{2x}\cos 3x + \frac{1}{3}e^{2x}\sin 3x + \frac{2}{10}$$

$[\sin 3x + 3\cos 3$

Q No 4

Solve $(D^2 - DD')z = \cos x \cos 2y$

Solution:

The equation is in symbolic form

$(D^2 - DD')z = \cos x \cos 2y$ — (A)

Put A.E $D^2 - DD' = 0$

We know that

$\frac{D}{D'} = m$ i.e $D = m, D' = 1$

$\Rightarrow m^2 - m = 0$

$m = 0, 1$

Therefore $C.F = f_1(y) + f_2(y+x)$

From eq (A)

$P.I = \frac{1}{D^2 - DD'} \cos x \cos 2y$

$= \frac{1}{2} \cdot \frac{1}{D^2 - DD'} 2\cos x \cos 2y$

As
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$C.F = f_1(y-x) + x f_2(y-x)$$

$$PI = \frac{1}{D^2+2DD'+D'^2}\left[2(y-x)+\sin(x-y)\right]$$

$$= \frac{1}{(D+D')^2}\left[2(y-x)+\sin(x-y)\right]$$

General method

$m_2 - 1$; $y - x = c$

$$= \frac{1}{D+D'}\int\left[2c+\sin(-c)\right]dx$$

$$= \frac{1}{D+D'}\left[2cx-(\sin c)x\right]$$

Putting $c = y - x$

$$= \frac{1}{D+D'}\left[2x(y-x) - x\sin(y-x)\right]$$

Putting again $y - x = c$

$= \int (2xc - x\sin c)\,dx$

$\Rightarrow cx^2 - \frac{x^2}{2}\sin c$

putting $c = y - x$

$= x^2(y-x) - \frac{x^2}{2}\sin(y-x)$

$= x^2y - x^3 + \frac{x^2}{2}\sin(x-y)$

Hence the required Solution is

$z = C.F + P.I$

$z = f_1(y-x) + x f_2(y-x) + x^2y - x^3 + \frac{1}{2}x^2$ $\sin(x-y)$