QUESTION No. 1

i. $W = \sin(x+ct) + \cos(2x+2ct)$

Solution

$$\frac{\partial w}{\partial t} = \cos(x+ct)+c - \sin(2x-2ct)+2c$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct)+c^2 - \cos(2x+2ct)+4c^2 \quad \text{—(1)}$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - \sin(2x+2ct)+2$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

$$= [-\sin(x+ct) - 4\cos(x+2ct)]$$

$$\frac{\partial^2 w}{\partial^2 t^2} = +c^2[-\sin(x+ct) - 4(\cos(2x+2ct)]$$

$$c^2 = \frac{\partial^2 w}{\partial x^2}$$

ii. $w = \tan(2x+ct)$

Solution:

$$\frac{\partial w}{\partial t} = \sec^2(2x+ct)\frac{\partial}{\partial t}(2x+ct)$$

$$= c\sec^2(2x+ct)$$

$$\frac{\partial w^2}{\partial t^2} = c\,2\,Sec(2x+ct)\,Sec(2x+ct)\tan(2x+ct)$$

$$\frac{\partial w^2}{\partial t^2} = 2c^2 Sec^2(2x+ct)$$

$$\frac{\partial w}{\partial t^2} = 2\cdot 2Sec(2x+ct)\cdot Sec(2x+ct)\cdot\tan(2x+ct)\times 2$$

$$= c^2\,8\,sec^2(2x+ct)\tan(2x+ct)$$

$$= 2c^2 Sec^2(2x+ct)\tan(2x+ct) \neq c^2\,8\,sec^2(2x+ct)\tan(2x+ct)$$

So, it is not possible.
Not satisfied.

QUESTION No. 2

Given function is

$$F(x) = \begin{cases} x\,; & -\pi < x \leq 0 \\ 2x\,; & 0 \leq x \leq \pi \end{cases}$$

We have to find the Fourier co-efficients, a_0, a_n and b_n.

Now.

$$a_0 = \frac{1}{\pi}\int_{-\pi} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{\pi^{\cancel{2}}}{2}\right] + \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^{\cancel{2}}}{\cancel{2}} - 0\right]$$

$$\boxed{a_0 = -\frac{\pi}{2} + \pi = \pi/2} \quad \text{— (1)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (x\cos nx\,dx) + \frac{1}{\pi}\int_{0}^{\pi} (2x\cos nx)\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+ \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1 - (-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

So,

$$\boxed{a_n = \begin{cases} \dfrac{-2}{\pi n^2} ; & \text{if "}n\text{" is add} \\ 0 ; & \text{if "}n\text{" is even} \end{cases}} \quad \text{— (2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,\sin nx\,dx + \frac{2}{\pi}\int_{0}^{\pi} x\,\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$\boxed{b_n = \frac{1}{\pi}\left[-\pi\frac{\cos n\pi}{n}\right] + \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] = -3\frac{\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n}}$$

So the required fourier series is;

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n \sin nx\right)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

QUESTION No-3

$$y'' - 4y + 13y = 8\sin 3x \;,\; y(0) = 1 \text{ and } y(0) = 2.$$

Solution:

Associated homogeneous Eq of ① is,

$$y'' - 4y' + 13y = 0 \quad -②$$

To change ② into auxillary equation;

put $y = m$ in eq ②

$$m^2 - 4m + 13 = 0$$

Use Quadratic Formula

$$a = 1,\; b = -4,\; c = 13$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}.$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm \sqrt{36}\, i}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

$m_1 = 2 + 3i$

$m_2 = 2 - 3i$

$y_c = e^{2x}(C_1 \cos 3x + C_2 \sin 3x) \Rightarrow$ (A)

Let

$y_p = A \cos 3x + B \sin 3x \Rightarrow$ (x)

Diff. with respect to "x"

$y_p' = -3A \sin 3x + 3B \cos 3x$

Again, diff. w.r.t "x"

$y_p'' = -9A \cos 3x - 9B \sin 3x$

Put in (1)

$\Rightarrow (-9A \cos 3x - 9B \sin 3x) - 4(-3A \sin 3x + 3B \cos 3x) + 13(A \cos 3x + B \sin 3x) = 8 \sin 3x$

$\Rightarrow -9A \cos 3x - 12B \cos 3x + 13A \cos 3x - 9B \sin 3x + 12A \sin 3x + 13B \sin 3x = 8 \sin 3x$

$\Rightarrow (-9A - 12B + 13A) \cos 3x + (-9B + 12A + 13B) \sin 3x = 8 \sin 3x$

$\Rightarrow (4A - 12B) \cos 3x + (4B + 12A) \sin 3x = 8 \sin 3x$

Comparing co-efficients

$\sin 3x \Rightarrow 4B + 12A = 8 \Rightarrow (a)$

$\cos 3x \Rightarrow 4A - 12B = 0 \Rightarrow 4A = 12B$

$\Rightarrow \boxed{A = 3B}$ —(b)

Put (b) in eq (A)

$4B + 12(3B) = 8$

$4B + 36B = 8$

$40B = 8$

$B = 1/5$ —(c)

Put (c) in (b)

$\Rightarrow \boxed{A = 3/5} \Rightarrow (d)$

Put (c) and (d) in (x)

$y_p = 3/5 \cos 3x + 1/5 \sin 3x$ —(B)

The G. sol is

$y = y_c + y_p$

$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + 3/5 \cos 3x + \frac{1}{5} \sin 3x$ —(C)

Now we need to find the values of C_1 and C_2 for this

put $x=0$ and $y=1$ in (C)

$$1 = e^{x(2)}\left(C_1 \cos 3(0) + C_2 \sin 3(0) + \frac{3}{5} \cos 3(0) + \frac{1}{5} \sin 3(0)\right)$$

$$1 = \left(C_1 (1) + C_2(0)\right) + \frac{3}{5}(1) + \frac{1}{5}(0)$$

$$1 = C_1 + 3/5$$

$$C_1 = 1 - 3/5$$

$$\boxed{C_1 = 2/5} \Rightarrow (xx)$$

Diff. (C) w.r.t

$$y' = c_1\left(2e^{2x} \cos 3x - 3e^{2x} \sin 3x\right) + c_2\left(2e^{2x} \sin 3x + 3e^{2x} \cos 3x\right) - \frac{6}{5} \sin 3x + \frac{3}{5} \cos 3x \rightarrow (D)$$

put $y' = 2$, $x = 0$ in eq (D)

$$2 = c_1\left(2e^{2(0)}\cos 3(0) - 3e^{2(0)}\sin 3(0)\right) + c_2 \sin 3(0) + 3e^{2(0)}\cos 3(0) - \tfrac{6}{5}\sin 3(0) + \tfrac{3}{5}\cos 3(0)$$

$$2 = C_1(2) + C_2(3) - 0 + 3/5$$

$$2 = 2C_1 + 3C_2 + 3/5$$

Put $C_1 = 2/5$

$$2 = \frac{4}{5} + 3C_2 + 3/5$$

$$2 = \frac{7}{5} + 3C_2$$

$$3C_2 = 2 - 7/5$$

$$3C_2 = 3/5$$

$$\boxed{C_2 = \frac{3}{15}} \Rightarrow (xxx)$$

Put (xx) and (xxx) in (C)

$$y = e^{2x}\left(2/5 \cos 3x + 3/15 \sin 3x\right) + \frac{3}{5}\cos 3x + 1/5 \sin 3x$$

$$y = 2/5\, e^{2x}\cos 3x + \frac{3}{15} e^{2x}\sin 3x + 3/5 \cos 3x + 1/5 \sin 3x$$

QUESTION No. 4

$$(D^2 - DD')z = \cos x \cos 2y$$

The given PDE can be re-written as;

$$D(D-D')z = \cos x \cos 2y$$

Its CF is given by

$$CF = \phi_1(y) + \phi_2(y+x)$$

While its PI is given by:

$$PI = \frac{1}{(D^2 - DD')} \cdot \tfrac{1}{2}\left[\cos(x+2y) + \cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{(-1+2)}\cos(x+2y) + \frac{1}{(-1-2)}\cos(x-2y)\right]$$

$$= \frac{1}{2}\cos(x+2y - \tfrac{1}{6}\cos(x-2y)]$$

Hence the complete solution of the given PDE is given by

$$z = \phi_1(y) + (\phi_2)(y+x) + \tfrac{1}{2}\cos(x+2y) - \tfrac{1}{6}\cos(x-2y) \Rightarrow \text{Ans.}$$