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Spring Semester.

Final term

Differential Equations

Q1 a) Define second order linear homogeneous / Non-homogeneous along with example?

Ans: The general nth-order differential equation is of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_{n-1} \frac{dy}{dx} + P_n y = X \longrightarrow (1)$$

where $(P_0 \neq 0)$

where X and the co-efficients $P_0, P_1, P_2, \ldots P_n$ are the constants or the function of x.

Now if X is zero (identically) then the linear equation (1) is said to homogeneous.

And if the $X \neq 0$, then the linear equation (1) is said to be non-homogeneous differential equation.

For example

$$4x^3 y''' - 3x^2 y'' + 6xy' + 9y = \sin x$$

is the non-homogeneous differential equation of order three.

Now the equation

$$4y'' + 6y' - 8y = 4x^2$$

is said to be non-homogeneous differential equation of order two

Now the equation

$$3y'' - 2y' + 8y = 0$$

is said to homogeneous differential equation of order two.

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Q1

(i) $4y'' - 6y' + 7y = 0$

sol:-

$$4\frac{D^2y}{dx^2} - 6\frac{dy}{dx} + 7y = 0$$

$$\Rightarrow 4D^2y - 6Dy + 7y = 0$$

$$\Rightarrow (4D^2 - 6D + 7)y = 0$$

$$\Rightarrow F(D)y = 0$$

Now the auxilary equation is obtained by putting D by m in F(D) and put equall to zero.

i.e $F(m) = 0$

$$\Rightarrow 4m^2 - 6m + 7 = 0$$

using the Quadratic formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{6 \pm \sqrt{36 - 4(4)(7)}}{2(4)}$$

$$m = \frac{6 \pm \sqrt{-76}}{8}$$

$$\Rightarrow m = \frac{6 \pm i\sqrt{4 \times 19}}{8}$$

$$\Rightarrow m = \frac{6 \pm 2i\sqrt{19}}{8}$$

So $m = \dfrac{3 \pm i\sqrt{19}}{4}$

we have $m_1 = \dfrac{3 - i\sqrt{19}}{4}$ and $m_2 = \dfrac{3 + i\sqrt{19}}{4}$

the general solution will be

$$\cancel{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

$$\cancel{\Rightarrow y = c_1 e}$$

$$y = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$$

$$y = e^{3/4\, x}\left[c_1 \cos\left(\sqrt{19}/4\right)x + c_2 \sin\left(\sqrt{19}/4\right)x\right]$$

Ans

(ii) $y'' - 4y' - 12y = 3e^{5x}$

Sol:

$$\cancel{D^2y} \quad \frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = 3e^{5x}$$

$$\Rightarrow D^2y - 4Dy - 12y = 3e^{5x}$$

$$\Rightarrow (D^2 - 4D - 12)y = 3e^{5x}$$

$$\Rightarrow F(D)y = f(x)$$

let $y = y_c + y_p \longrightarrow *$

is the required general solution.

now for y_c, the auxilary equation will be

$$F(m) = 0$$

$$\Rightarrow m^2 - 4m - 12 = 0$$

$$\Rightarrow m^2 + 2m - 6m - 12 = 0$$

$$\Rightarrow m(m+2) - 6(m+2) = 0$$

$$\Rightarrow (m+2)(m-6) = 0$$

so

$m + 2 = 0$ or $m - 6 = 0$

$\Rightarrow m = -2$ and $m = 6$

so $m_1 = -2$ and $m_2 = 6$

so the solution is

$$y_c = c_1e^{m_1x} + c_2e^{m_2x}$$

$$\Rightarrow \boxed{y_c = c_1e^{-2x} + c_2e^{6x}} \longrightarrow (a)$$

Now for y_p

$$y_p = \frac{1}{F(D)} f(x)$$

$$\Rightarrow y_p = \frac{1}{D^2 - 4D - 12} 3e^{5x}$$

Replace D by $a = 5$ we have

$$y_p = \frac{1}{(5)^2 - 4(5) - 12} 3e^{5x}$$

$$\Rightarrow y_p = \frac{1}{25 - 20 - 12} 3e^{5x}$$

$$\Rightarrow \boxed{y_p = \frac{-3}{7} e^{5x}} \rightarrow (b)$$

Now putting (a) and (b) in (*)
we get

$$y = y_c + y_p$$

$$\Rightarrow \boxed{y = c_1 e^{-2x} + c_2 e^{6x} - \frac{3}{7} e^{5x}}$$

is the required general solution.

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Q2 Solve the following IVP for the 2nd order linear equation.

(i) $16y'' - 40y' + 25y = 0$ $y(0) = 3$ $y'(0) = -9/4$

sol:-

$$16\frac{d^2y}{dx^2} - 40\frac{dy}{dx} + 25y = 0$$

$$\Rightarrow 16D^2y - 40Dy + 25y = 0$$

$$\Rightarrow (16D^2 - 40D + 25)y = 0$$

$$\Rightarrow F(D) = 0$$

now the auxilary equation is obtained, by replacing m by D in F(D) and put it equall to zero. i-e

$$F(m) = 0$$

$$\Rightarrow 16m^2 - 40m + 25 = 0$$

$$\Rightarrow 16m^2 - 20m - 20m + 25 = 0$$

$$\Rightarrow 4m(4m-5) - 5(4m-5) = 0$$

$$\Rightarrow (4m-5)(4m-5) = 0$$

so

$$4m - 5 = 0 \Rightarrow m = 5/4$$

Hence

$$m_1 = 5/4$$

$$m_2 = 5/4$$

So the roots are real and same

Hence the solution is

$$y = (c_1 + c_2 x) e^{5/4 x} \longrightarrow (*)$$

Now for $y(0) = 3$

$\Rightarrow x = 0 \quad y = 3$

put in $(*)$ we have

$$y = (c_1 + c_2(0)) e^{5/4(0)}$$

$\Rightarrow 3 = c_1$

$\Rightarrow \boxed{c_1 = 3}$

Now Differentiate eq $(*)$ w.r.t "x"

$$y' = (c_1 + c_2 x) e^{5/4 x} (5/4) + e^{5/4 x} c_2$$

now put $y'(0) = -9/4$

$\Rightarrow x = 0 \quad y' = -9/4$

we have

$$-9/4 = (c_1 + c_2(0)) e^{5/4(0)} (5/4) + e^{0} c_2$$

$$\Rightarrow -9/4 = c_1 \frac{5}{4} + c_2$$

put $c_1 = 3$ we have

$$-9/4 = 3(5/4) + c_2$$

$$\Rightarrow -\frac{9}{4} = \frac{15}{4} + c_2$$

$$\Rightarrow c_2 = -\frac{9}{4} - \frac{15}{4}$$

$$\Rightarrow c_2 = -\left(\frac{9}{4} + \frac{15}{4}\right)$$

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$$c_2 = -\left(\frac{9}{4} + \frac{15}{4}\right)$$

$$\Rightarrow c_2 = -\left(\frac{24}{4}\right)$$

$$\Rightarrow \boxed{c_2 = -6}$$

So putting the values of $c_1 = 3$ and $c_2 = -6$ in eq ⊛ we have

$$y = (c_1 + c_2 x)\, e^{5/4\, x}$$

$$\Rightarrow \boxed{y = (3 - 6x)\, e^{5/4\, x}}$$ Ans

is the required particular Solution.

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(ii) $y'' + 14y' + 49y = 0$ $\quad y(-4) = -1$

$y'(-4) = 5$

Sol:

$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0$$

$$\Rightarrow D^2y + 14Dy + 49y = 0$$

$$\Rightarrow (D^2 + 14D + 49)y = 0$$

$$\Rightarrow F(D)y = 0$$

now the auxilary equation will be, replace D by m in F(D)

i.e

$$F(m) = 0$$

$$\Rightarrow m^2 + 14m + 49 = 0$$

$$\Rightarrow m^2 + 7m + 7m + 49 = 0$$

$$\Rightarrow m(m+7) + 7(m+7) = 0$$

$$\Rightarrow (m+7)(m+7) = 0$$

$$\Rightarrow m = -7 \text{ and } m = -7$$

so $m_1 = -7$, $m_2 = -7$

so the general solution is

$$\boxed{y = (c_1 + c_2x)e^{-7x}} \longrightarrow (*)$$

Now put $y(-4) = -1$

$$\Rightarrow x = -4 \text{ and } y = -1$$

so we have

$$-1 = (c_1 - 4c_2)e^{-7(-4)}$$

$$\Rightarrow -1 = (c_1 - 4c_2)e^{28} \longrightarrow (1)$$

Now Differentiate (*) with respect to "x"

So we have

$$y' = (c_1 + c_2 x)e^{-7x}(-7) + e^{-7x} c_2 \longrightarrow (2)$$

put $y'(-4) = 5$

$\Rightarrow x = -4$ and $y' = 5$ put in (2)

so we have

$$5 = (c_1 - 4c_2)e^{28}(-7) + e^{28} c_2$$

$$\Rightarrow 5 = (c_1 - 4c_2 + c_2)e^{28}$$

$$\Rightarrow 5 = (c_1 - 3c_2)e^{28} \longrightarrow (3)$$

Now solving (1) and (3)

$$-1 = c_1 e^{28} - 4c_2 e^{28}$$

$$\underline{\pm 5 = \cancel{c_1 e^{28}} \mp 3c_2 e^{28}}$$

$$-6 = -c_2 e^{28}$$

$$\Rightarrow 6 = c_2 e^{28}$$

$$\Rightarrow \boxed{c_2 = \frac{6}{e^{28}}} \text{ put in } (1)$$

$$-1 = \left(c_1 - 4\left(\frac{6}{e^{28}}\right)\right)e^{28}$$

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$\Rightarrow -1 = c_1 e^{28} - 24$

$\Rightarrow -1 + 24 = c_1 e^{28}$

$\Rightarrow c_1 e^{28} = 23$

$\Rightarrow \boxed{c_1 = \frac{23}{e^{28}}}$

Putting the values of c_1 & c_2 in eq (*)

$$y = \left[\frac{23}{e^{28}} + \frac{6}{e^{28}} x\right] e^{-7x}$$

is the required Solution.

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(iii) $y'' - 4y' + 9y = 0$ $\quad y(0) = 0 \quad y'(0) = -8$

Sol:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 9y = 0$$

$$\Rightarrow D^2y - 4Dy + 9y = 0$$

$$\Rightarrow (D^2 - 4D + 9)y = 0$$

$$\Rightarrow F(D)y = 0$$

Now the auxilary equation is

$$F(m) = 0$$

$$\Rightarrow m^2 - 4m + 9 = 0$$

using Quadratic formula.

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow m = \frac{4 \pm \sqrt{16 - 4(1)(9)}}{2}$$

$$\Rightarrow m = \frac{4 \pm \sqrt{-20}}{2}$$

$$\Rightarrow m = \frac{4 \pm i\sqrt{4 \times 5}}{2}$$

$$\Rightarrow m = \frac{+4 \pm 2i\sqrt{5}}{2}$$

$$\Rightarrow m = \frac{2(2 \pm i\sqrt{5})}{2}$$

$$\Rightarrow m = 2 \pm i\sqrt{5}$$

So the roots are complex and distinct

So the general solution is

$$y = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]$$

$$\Rightarrow y = e^{2x}\left[c_1 \cos(\sqrt{5}\,x) + c_2 \sin(\sqrt{5}\,x)\right] \longrightarrow (*)$$

now for $y(0) = 0$

put $x = 0$ and $y = 0$ in $(*)$

$$y = e^{0}\left[c_1 \cos(0) + c_2 \sin(0)\right]$$

$$\Rightarrow 0 = c_1(1)$$

$$\Rightarrow \boxed{c_1 = 0}$$

now Differentiate $(*)$ w.r.t "x"

we have

$$y' = e^{2x}\left[c_1(-\sin\sqrt{5}x)(\sqrt{5}) + c_2 \cos(\sqrt{5}\,x)(\sqrt{5})\right]$$
$$+ \left(c_1 \cos(\sqrt{5}x) + c_2 \sin(\sqrt{5}x)\right) e^{2x}(2)$$

put $x = 0$ and $y' = -8$

we have

$$-8 = e^{0}\left[c_1(0) + c_2 \cos(0)\sqrt{5}\right] + \left(c_1 \cos(0) + 0\right) e^{0}$$

$$\Rightarrow \boxed{-8 = c_2\sqrt{5} + c_1} \rightarrow (1)$$

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put $c_1 = 0$ in eq ①

$$-8 = c_2\sqrt{5}$$

$$\Rightarrow \quad c_2 = -8/\sqrt{5}$$

putting the values of c_1 & c_2 in eq ⊛

$$y = e^{2x}\left[0 + \frac{-8}{\sqrt{5}}\sin\sqrt{5}\,x\right.$$

$$\Rightarrow y = e^{2x}\left(-\frac{8}{\sqrt{5}}\sin\sqrt{5}\,x\right)$$

is the required solution.

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(iv) $y'' - 8y' + 17y = 0 \quad y(0) = -4 \quad y'(0) = -1$

Sol:-

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 17y = 0$$

$$\Rightarrow D^2y - 8Dy + 17y = 0$$

$$\Rightarrow (D^2 - 8D + 17)y = 0$$

$$\Rightarrow F(D)y = 0$$

now the axuilary equation

i-e $F(m) = 0$

$$\Rightarrow m^2 - 8m + 17 = 0$$

using Quadratic formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow m = \frac{8 \pm \sqrt{64 - 4(1)(17)}}{2}$$

$$\Rightarrow m = \frac{8 \pm \sqrt{64 - 68}}{2}$$

$$\Rightarrow m = \frac{8 \pm \sqrt{-4}}{2}$$

$$\Rightarrow m = \frac{8 \pm i\sqrt{2}}{2}$$

$$\Rightarrow m = \frac{8}{2} \pm \frac{i\sqrt{2}}{2}$$

so $m = \frac{4 \pm i\sqrt{2}}{2}$

$\Rightarrow m = \frac{4 \pm i\sqrt{2}}{2}$

so $m_1 = \frac{4 - i\sqrt{2}}{2}$ $m_2 = \frac{4 + i\sqrt{2}}{2}$

so the general solution is

$y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

$\Rightarrow y = e^{4x}\left[c_1 \cos\left(\frac{\sqrt{2}}{2}x\right) + c_2 \sin\left(\sqrt{2}/2\, x\right)\right] \longrightarrow (*)$

Now for $y(0) = -4$

$x = 0$ and $y = -4$ put in $(*)$

$-4 = e^{0}[c_1 \cos(0) + c_2 \sin(0)]$

$\Rightarrow \boxed{-4 = c_1}$

Now Differentiate $(*)$ w.r.t "x"

we have

$y' = e^{4x}\left[c_1\left(-\sin\frac{\sqrt{2}}{2}x\right)(\sqrt{2}/2) + c_2 \cos\left(\frac{\sqrt{2}}{2}x\right)(\sqrt{2}/2)\right]$

$+ \left(c_1 \cos\left(\frac{\sqrt{2}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{2}}{2}x\right)\right) e^{4x}(4)$

Now put $y'(0) = -1$

$\Rightarrow x = 0$ and $y' = -1$

$$-1 = e^{0}\left[c_1(0) + c_2 \cos(0)\left(\frac{\sqrt{2}}{2}\right)\right]$$

$$+\left(c_1 \cos(0) + c_2 \sin(0)\right) e^{0}(4)$$

$$\Rightarrow -1 = \left(c_2 \frac{\sqrt{2}}{2}\right) + (c_1 + 0)4$$

$$\Rightarrow -1 = c_2 \frac{\sqrt{2}}{2} + 4c_1$$

$\Rightarrow -2 = \sqrt{2}\, c_2 + 8c_1$ put $c_1 = -4$

we have

$$-2 = \sqrt{2}\, c_2 + 8(-4)$$

$$\Rightarrow -2 = \sqrt{2}\, c_2 - 32$$

$$\Rightarrow \sqrt{2}\, c_2 = -2 + 32$$

$$\Rightarrow \sqrt{2}\, c_2 = 30$$

$$\Rightarrow \boxed{c_2 = \frac{30}{\sqrt{2}}}$$

Putting the values of c_1 & c_2 in (*)

$$y = e^{4x}\left[-4\cos\left(\frac{\sqrt{2}}{2}x\right) + \frac{30}{\sqrt{2}} \sin\left(\frac{\sqrt{2}}{2}x\right)\right]$$

Ans

Q3 A Find the laplace transform along with example?

① $f(t) = 6(e^{-5t}) + e^{3t} + 5(t^3) - 9$

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} - \frac{30}{s^4} - \frac{9}{s}$$

$$= \frac{6}{10} + \frac{1}{2} - \frac{30}{5^4} - \frac{9}{5} \text{ Ans}$$

② $g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

$$G(s) = 4\frac{s}{s^2+(4)^2} - 9\frac{4}{s^2+(4)^2} + 2\frac{s}{s^2+(10)^2}$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} - \frac{2s}{s^2+100} \text{ Ans}$$

iii) $h(t) = e^{3t} + \cos(6t) - e^{(3t)}\cos(6t)$

$$H(s) = \frac{1}{s-3} + \frac{s}{s^2+(6)^2} - \frac{s-3}{(s-3)^2+(6)^2}$$

$$= \frac{1}{s-3} - \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

Q#4 Solve the following IVP using Laplace Transform.

i) $y'' - 10y' + 9y = 5t, \quad y(0) = -1, \; y'(0) = 2$

$$\mathcal{L}\{y''\} - 10\,\mathcal{L}\{y'\} + 9\,\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 10(sY)(s) - y(0))$$

$$+ 9Y(s) = \frac{5}{s^2}$$

$$(s^2 - 10s + 9)Y(s) + 5 - 12 = \frac{5}{s^2}$$

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-5}{(s-9)(s-1)}$$

$$Y(s) = \frac{5+12s^2-s^2}{s^2(s-9)(s-1)}$$

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

$5+12s^2-s^2 = As(s-9)(s-1)$

$B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$

$s=0 \quad , \quad 5 = 9B \quad \Rightarrow B = 5/9$

$s=1 \quad , \quad 16 = -8D \quad \Rightarrow D = -2$

$s=9 \quad , \quad 248 = 648C \quad \Rightarrow C = \frac{31}{81}$

$s=2 \quad 45 = -14A = \frac{4345}{81} \quad \Rightarrow A = \frac{50}{81}$

$$Y(s) = \frac{50}{80} + \frac{5/9}{s^2} + \frac{31/81}{s-9} - \frac{2}{s-1}$$

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{at} - 2e^{t} \text{ Ac}$$

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ii) $y'' - 6y' + 15y = 2\sin(3t)$, $y(0) = 1$ & $y'(0) = -4$

$s^2Y(s) - sY(0) - 6(sY(s) - y(0)) + 15Y(s) = 2\frac{3}{s^2+9}$

$(s^2 - 6s + 15)Y(s) + s - 2 = \frac{6}{s^2+9}$

$Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)}$

$Y(s) = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+6s+15}$

$-s^3 + 2s^2 + 9s + 24 = (As+B)(s^2 - 6s + 15) + (Cs+D)$

$+ (Cs+D)(s^2+9)$

$= (A+C)s^3 + (-6A+B+D)s^2 + (15A - 6B + 9C)s$

$+ 15B + 9D.$

$s^3 = A + C = -1 \Rightarrow A = \frac{1}{10}$

$s^2 = -6A + B + D = 2 = B = \frac{1}{10}$

$s' = 15B + 9D = 24 = D = \frac{5}{2}$

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$$Y(s) = \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s+25}{s^2-6s+15}\right)$$

$$= \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11(s-3+3)+25}{(s-3)^2+6}\right)$$

$$= \frac{1}{10}\left(\frac{s}{s^2+9} + \frac{\frac{1}{3}\cdot 3}{s^2+9} - \frac{11(s-3)}{(s-3)^2+6} - \frac{\frac{8\sqrt{6}}{\sqrt{6}}}{(s-3)^2+6}\right)$$

$$y(t) = \frac{1}{10}\left(\cos(3t) + \frac{1}{3}\sin(3t) - 11e^{3t}\cos(\sqrt{6}t) - \frac{8}{\sqrt{6}}e^{3t}\sin(\sqrt{6}t)\right)$$