Q1:- Let $P(1,0,-3)$, $Q(0,-2,-4)$ and $R(4,1,6)$ be points

a) Find the equation of the plane through points P, Q and R.

Solution:-

Since equation of plane through given points is a

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0 \quad \text{— (i)}$$

where (x_0, y_0, z_0) is any point on plane and $\langle a,b,c\rangle$ is perpendicular to the plane.

Thus

$$\vec{PQ} = \langle 0-1, -2-0, -4+3\rangle$$
$$= \langle -1, -2, -1\rangle$$

and

$$\vec{PR} = \langle 4-1, 1-0, 6+3\rangle$$
$$= \langle 3, 1, 9\rangle$$

Now we take cross product of above two vectors

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & -1 \\ 3 & 1 & 9 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -2 & -1 \\ 1 & 9 \end{vmatrix} - \hat{j}\begin{vmatrix} -1 & -1 \\ 3 & 9 \end{vmatrix} + \hat{k}\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix}$$

$$= \hat{i}(-18+1) - \hat{j}(-9+3) + \hat{k}(-1+6)$$

$$= \hat{i}(-17) - \hat{j}(-6) + \hat{k}(5)$$

$$= -17\hat{i} + 6\hat{j} + 5\hat{k}$$

Let P be any point on plane, so putting values in eq (i)

$$-17(x-1) + (6)(y-0) + 5(z-(-3)) = 0$$

$$-17(x-1) + 6(y) + 5(z+3) = 0$$

$$-17x + 17 + 6y + 5z + 15 = 0$$
$$-17x + 6y + 5z + 32 = 0$$
$$-17x + 6y + 5z = -32$$

Multiplying both sides by (-1)

$$17x - 6y + 5z = 32.$$

which is required eq. of plane through given three points.

b) Find the area of triangle with vertices P, Q & R.

Solution:-

Since area of triangle through given vertices is

$$\text{area of } \Delta = \frac{|\vec{PQ} \times \vec{PR}|}{2} \quad \text{— (i)}$$

As $\vec{PQ} = \langle -1, -2, -1 \rangle$ and $\vec{PR} = \langle 3, 1, 9 \rangle$

their cross product is

$$|\vec{PQ} \times \vec{PR}| = \langle -17, 6, 5 \rangle$$

Putting values in (i)

$$\text{area of } \Delta = \frac{\sqrt{(-17)^2 + (6)^2 + (5)^2}}{2}$$
$$= \sqrt{289 + 36 + 25} / 2$$
$$= \sqrt{350} / 2$$

$$\text{area of } \Delta = \frac{5\sqrt{14}}{2} \text{ units}.$$

Q2:- Let $f(x,y) = (x-y)^3 + 2xy + x^2 - y$. Find the linear approximation $L(x,y)$ near the point $(1,2)$.

Solution:-

Linear approximation $L(x,y)$ near the given point is

$$L(x,y) = f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0) \quad \text{---(i)}$$

As given $f(x,y) = (x-y)^3 + 2xy + x^2 - y$

and $P(x_0,y_0) = P(1,2)$

$$f(x,y) = (x-y)^3 + 2xy + x^2 - y$$

$$f(1,2) = (1-2)^3 + 2(1)(2) + (1)^2 - (2)$$

$$= (-1)^3 + 4 + 1 - 2 = \cancel{-1} + 4 + \cancel{1} - 2$$

$$= 2$$

Now we find derivatives w.r.t x and y.

$$f_x(x,y) = 3(x-y)^2 + 2y + 2x$$

$$f_x(1,2) = 3(1-2)^2 + 2(2) + 2(1)$$

$$= 3(-1)^2 + 4 + 2 = +3 + 4 + 2 = 7 + 2$$

$$= 9$$

and

$$f_y(x,y) = 3(x-y)^2(-1) + 2x - 1$$

$$f_y(1,2) = 3(1-2)^2(-1) + 2(1) - 1$$

$$= 3(-1)^2(-1) + 2 - 1 = -3 + 2 - 1$$

$$= -2$$

Putting values in eq (i)

$$L(x,y) = f(1,2) + f_x(x-1) + f_y(y-2)$$

Let (x,y) be $(1.1, 1.9)$, then

P.T.O

$\Rightarrow$ Q2

$= 2 + 9(x-1) + (-2)(y-2)$

$= 2 + 9(1.1-1) - 2(1.9-2)$

$= 2 + 9(0.1) - 2(-0.1)$

$= 2 + 0.9 + 0.2$

$= 3.1.$

Q3:- Find the distance between parallel planes $x+2y-z=-1$ and $3x+6y-3z=3$.

Solution:-

Given that

$x+2y-z=-1$ —— (i)

and $3x+6y-3z=3$ —— (ii) are two parallel planes.

Formula for distance between parallel planes is

$$d = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Put $y=z=0$ in eq (i), we get

$x=-1$

Thus $P(x_1, y_1, z_1) = P(-1,0,0)$ is a point on plane (i)

Now we find perpendicular distance from point $P(-1,0,0)$ to plane (ii)

$$d = \frac{|3(-1)+6(0)+(-3)(0)+(-3)|}{\sqrt{(3)^2+(6)^2+(-3)^2}} = \frac{|-3-3|}{\sqrt{9+36+9}}$$

$$= \frac{|-6|}{\sqrt{54}} = \frac{6}{\sqrt{54}}$$

Q4:- Find the following limit, if it exist, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy + y^2}{x^2+y^2}$$

Solution:-

Given that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy + y^2}{x^2+y^2}$

$\boxed{x\text{-axis}}$ limit along the path $y=0$

$$= \lim_{(x,y)\to(0,0)} \left.\frac{x^2 - xy + y^2}{x^2+y^2}\right|_{y=0}$$

$$= \lim_{(x,0)\to(0,0)} \frac{x^2 - x(0) + (0)^2}{x^2 - (0)^2} = \lim_{(x,0)\to(0,0)} \frac{\cancel{x^2}}{\cancel{x^2}} = \underline{1}$$

$\boxed{y\text{-axis}}$ limit along the path $x=0$

$$= \lim_{(x,y)\to(0,0)} \left.\frac{x^2 - xy + y^2}{x^2+y^2}\right|_{x=0}$$

$$= \lim_{(0,y)\to(0,0)} \frac{(0)^2 - (0)(y) + (y)^2}{(0)^2 + (y)^2} = \lim_{(0,y)\to(0,0)} \frac{y^2}{y^2} = 1$$

limit along $y = x$

$$= \lim_{(x,y)\to(0,0)} \left.\frac{x^2 - xy + y^2}{x^2+y^2}\right|_{y=x}$$

$$= \lim_{(x,x)\to(0,0)} \frac{x^2 - x\cdot x + x^2}{x^2+x^2} = \lim_{(x,x)\to(0,0)} \frac{\cancel{x^2} - \cancel{x^2} + x^2}{2x^2}$$

P.T.O →

⇒Q4 $= \frac{x^2}{2x^2} = \frac{1}{2}$

As we obtained different limits, so the limit of given $\lim_{(x,y)\to(0,0)} \frac{x^2 - xy + y^2}{x^2+y^2}$ does not exist.

Q5: Find the directional derivative of the function $f(x,y,z) = xyz$ in the direction of vector $v = \langle 5,-3,2 \rangle$

Solution :-

Given function $f(x,y,z) = xyz$

and vector $v = \langle 5,-3,2 \rangle$

Since directional derivative of the function is

$$DD = \nabla f \cdot \hat{v} \quad \text{—— (i)}$$

So first we find ∇f and $\hat{v}$

$$\nabla f = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$= yz\hat{i} + xz\hat{j} + xy\hat{k}$$

Now $\hat{v} = \frac{v}{|v|} = \frac{5\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{5^2 + (-3)^2 + (2)^2}} = \frac{5i - 3j + 2k}{\sqrt{25+9+4}}$

$$= \frac{5i - 3j + 2k}{\sqrt{38}}$$

Putting values in (i)

$$DD = (yz\hat{i} + xz\hat{j} + xy\hat{k})\left(\frac{5i - 3j + 2k}{\sqrt{38}}\right) = \frac{5yz - 3xz + 2xy}{\sqrt{38}}$$

Ans.

Q6:- Find the equation of tangent plane to the surface $z = 4x^3y^2 + 2y$ at point $(1, -2, 12)$

Solution:-

Given that

$$z = 4x^3y^2 + 2y$$

or

$$4x^3y^2 + 2y - z = 0$$

Since equation of tangent plane to the surface is given by $F(x,y,z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0)$$
$$+ F_z(x_0,y_0,z_0)(z-z_0) = 0 \quad \text{—— (a)}$$

Thus $F(x,y,z) = 4x^3y^2 + 2y - z = 0$

$$F_x = 12x^2y^2$$

$$F_x(1,-2,12) = 12(1)^2(-2)^2 = 12(1)(4) = 48$$

$$F_y = 8x^3y + 2$$

$$F_y(1,-2,12) = 8(1)^3(-2) + 2 = 8(-2) + 2 = -16 + 2 = -14$$

$$F_z = -1$$

$$F_z(1,-2,12) = -1$$

Now putting values in (a)

$$48(x-1) + (-14)(y-(-2)) + (-1)(z-12) = 0$$

$$48(x-1) - 14(y+2) - 1(z-12) = 0$$

$$48x - 48 - 14y - 28 - z + 12 = 0$$

$$48x - 14y - z - 64 = 0 \quad \text{or} \quad \boxed{48x - 14y - z = 64}$$

Q7:- Let $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ be any two vectors in space. Show the following identity that relates the cross product and the dot product $|u \times v|^2 + |u \cdot v|^2 = |u|^2 |v|^2$

Solution:-

Given two vectors

$u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$

and if θ is an angle between u and v, so

$$|u \times v| = |\bar{u}||\bar{v}| \sin\theta \, \hat{n}$$

$$|u \cdot v| = |\bar{u}||\bar{v}| \cos\theta$$

Since we have

$$|u \times v|^2 + |u \cdot v|^2 = |u|^2 |v|^2$$

So

$$= \left(|\bar{u}||\bar{v}| \sin\theta \right)^2 + \left(|\bar{u}||\bar{v}| \cos\theta \right)^2$$

$$= |u|^2 |v|^2 \sin^2\theta + |u|^2 |v|^2 \cos^2\theta$$

$$= |u|^2 |v|^2 \sin^2\theta + |u|^2 |v|^2 \cos^2\theta$$

$$= |u|^2 |v|^2 \left(\sin^2\theta + \cos^2\theta \right)$$

$$= |u|^2 |v|^2 (1) \qquad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= |u|^2 |v|^2$$

Q8:- What is the angle between the two planes $x+y=0$ and $y-z=2$?

Solution:-

Since angle between planes with normals is

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \|\vec{n}_2\|} \longrightarrow (a)$$

where n_1 and n_2 are normals vectors.

So

$$x+y=0 \quad \text{--- (i)}$$
$$y-z=0 \quad \text{--- (ii)}$$

Thus $\vec{n}_1 = \langle 1,1,0 \rangle$ and $\vec{n}_2 = \langle 0,1,-1 \rangle$

$$\vec{n}_1 \cdot \vec{n}_2 = (1\times 0) + (1\times 1) + (0\times -1)$$
$$= 0+1+0 = 1$$

$$\|\vec{n}_1\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$$
$$\|\vec{n}_2\| = \sqrt{0^2+1^2+(-1)^2} = \sqrt{2}$$

Putting values in (a)

$$\cos\theta = \frac{|1|}{(\sqrt{2})(\sqrt{2})} = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\theta = 60^\circ}$$