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Section:

A

Subject:

Differential Equations

Assignment #:

02

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Question # 01

Cauchy Euler Equation

① $x^3y''' + 2x^2y' + 2y = 10x + \frac{10}{x}$

Given.

$$x^3y''' + 2x^2y' + 2y = 10x + 10x^{-1}$$

Required:

To solve by Euler Cauchy Equation = ?

Solution:

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{dy}{dx} + 2y = 10x + 10x^{-1}$$

$$x^3D^3y + 2x^2D + 2y = 10x + 10x^{-1}$$

$$(x^3D^3 + 2x^2D + 2)y = 10x + 10x^{-1} \text{ — (}$$

Let $x = e^t$

$\Rightarrow t = \ln x$

$xD = D$

$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

$x^3D^3 = \Delta(\Delta-1)(\Delta-2)$

Putting into eq (1)

$(\Delta - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta) + 2)y = 10^x + 10x^{-1}$

$(\Delta^3 - \Delta^2 + 2)y = 10x + 10x^{-1}$

$(m^3 - m^2 + 2)y = 10e^t + \frac{10}{e^t}$

Using Synthetic division

$$\begin{array}{r|rrrr} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$\Delta^2 - 2\Delta + 2 = 0$

Using quadratic formula

$a = 1,\ b = -2,\ c = 2$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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$$D = \frac{-(-2) \pm \sqrt{-2^2 - 4(1)(2)}}{2a}$$

$$D = \frac{2 \pm \sqrt{4-8}}{2}$$

$$D = \frac{2 \pm \sqrt{-4}}{2}$$

$$D = \frac{2 \pm \sqrt{-1} \times \sqrt{4}}{2}$$

$$D = \frac{2 \pm 2i}{2}$$

$$D = \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

$$D = 1 \pm i$$

As roots are Complex, So

$$y_c = e^{-x}(c_1 \cos t + c_2 \sin t)$$

Now Particular integration

$$y_p = \frac{1}{D^3 - D^2 + 2} 10e^t + \frac{1}{D^3 - D^2 + 2} \frac{10}{e^t}$$

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$$y_p = \frac{1}{D^3-D^2+2} 10e^t + \frac{1}{D^3-D^2+2} \frac{10}{e^t}$$

$$= \frac{10e^t}{(1)^3-(1)^2+2} + \frac{10e^{-t}}{(1)^3-(1)^2+2}$$

$$= \frac{\cancel{10}^{5}e^t}{\cancel{2}} + \frac{\cancel{10}^{5}e^{-t}}{\cancel{2}}$$

$$= 5e^t + 5e^{-t}$$

General solution:

$$y = y_c + y_p$$

$$y = e^{-x}(c_1\cos t + c_2\sin t) + 5e^t + 5e^{-t}$$

putting $e^t = x$ and $t = \ln x$

$$\boxed{y = e^{-x}(c_1 \ln x + c_2 \sin x) + 5e^x + 5e^{-x}}$$

Ans

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Question #2

2) $x^3\frac{d^3y}{dx} + 4x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} - 15y = x^4$

Solution:

Let $\frac{d}{dx} = \theta$

$x^3D^3y + 4x^2D^2y - 5xDy - 15y = x^4$

$(x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4$

Let $x = e^t$

$\Rightarrow t = \ln x$

$xD = D$

$x^2D^2 = \theta(\theta-1) = \theta^2 - \theta$

$x^3D^3 = \theta(\theta-1)(\theta-2) = \theta^3 - 3\theta^2 + 2\theta$

Now Substituting

$(x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4$

$(\theta^3 - 3\theta^2 + 2\theta + 4(\theta^2 - \theta) - 5(\theta) - 15)y = e^{4t}$

$(D^3+D^2-7D-15)y = e^{4t}$

Synthetic division.

5	1	+1	-7	-15
		3	12	15
	1	4	5	0

$D^2+4D+5=0$

Quadratic formula

$$D = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-4\pm\sqrt{4^2-4(1)(5)}}{2}$$

$$= \frac{-4\pm 2i}{2}$$

$$D = \frac{\cancel{2}(-2\pm i)}{\cancel{2}}$$

$$y_c = e^{4t}(c_1 \cos t + c_2 \sin t)$$

For $y_p = ?$

$$y_p = \frac{1}{D^3 + D^2 - 7D - 15} \, e^{4t}$$

$$= \frac{1}{(4)^3 + (4^2) - 7(4) - 15} \, e^{4t}$$

$$= \frac{1}{64 + 16 - 28 - 15} \, e^{4t}$$

$$= \frac{1}{80 - 43} \, e^{4t}$$

$$y_p = \frac{1}{37} e^{4t}$$

Hence

$$y = y_c + y_p$$

$$y = (C_1 \cos t + C_2 \sin t) + \frac{1}{37} e^{4t}$$

again put $t = \ln x$ and $x = \ln x$

$$y = e^{3x}(C_1 \cos \ln x + C_2 \sin \ln x) + \frac{1}{37} e^{4x}$$

Ans.

Question # 03

$x^2y'' + 2xy' - 6y = 10x^2$

Solution:

$y(1) = 1$ and $y'(1) = -6$

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 6y = 10x^2$$

$$\Rightarrow \left(x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} - 6\right)y = 10x^2$$

put ~~θD~~ $xD = \theta$

$$\Rightarrow x^2D^2 = \theta(\theta-1) = \theta^2 - \theta$$

$x = e^t$ and $\log x = t$

$$(\theta^2 - \theta + 2\theta - 6)y = 10e^{2t}$$

$$(\theta^2 + \theta - 6)y = 10e^{2t}$$

The characteristic equation

$$\theta^2 + \theta - 6 = 0$$

$$\theta^2 + 3\theta - 2\theta - 6 = 0$$

$$\theta(\theta+3) - 2(\theta+3) = 0$$

$$(\theta+3)(\theta-2) = 0$$

$D = 2, \; D = -3$

As roots are real and distinct
For $y_c = ?$

$$y_c = c_1 e^{-3t} + c_2 e^{2t}$$

For $y_p = ?$

$$y_p = \frac{1}{D^2 - D - 6} \cdot 10^{2t}$$

$$y_p = \frac{10}{D^2 - D - 6} e^{2t}$$

$$y_p = 10 \cdot \frac{1}{D} e^{2t}$$

Now $10 \cdot \frac{1}{\frac{d}{d} D(D^2 + D - 6)} e^{2t}$

$$\Rightarrow 10 \frac{t}{2D+1} e^{2t}$$

$$= 10 \frac{1 \cdot t}{4+1} e^{2t}$$

$$y_p = 2te^{2t}$$

General Solution

$$y = y_c + y_p$$

$$= C_1 e^{-3t} + C_2 e^{2t} + 2te^{2t}$$

$$y = C_1 x^{-3} + C_2 x^2 + 2(\log x) x^2 \quad \text{—} \; (A)$$

put $y(1) = 1$, i.e $x = 1$; $y = 1$
in eq. (A)

$$1 = C_1(1)^{-6} + (2(1)^2 + 2\log(1))$$

$$1 = C_1 + C_2 \quad \text{—} \; (B)$$

Now differentiate eq (A) w.r.t x

$$y' = -3C_1 x^{-4} + 2(2x + \frac{2}{x}(x^2) + 4x\log x$$

Now put $y'(1) = -6$, i.e $y' = -6$
and $x = 1$

$$-6 = -3C_1 + 2C_2 + 2 + 0$$

$$\Rightarrow -6 - 2 = -3C_1 + 2C_2 = 0$$

$$\Rightarrow -8 = -3C_1 + 2C_2 \quad \text{—} \; (C)$$

Multiplying eq (B) with 2 and
Subtracting from (C)

$$2c_1 + \cancel{2c_2} = 2$$
$$\mp 3c_1 \pm \cancel{2c_2} = \mp 8$$

$$5c_1 = 10$$

$$c_1 = \frac{\cancel{10}^{2}}{\cancel{5}} = 2$$

$$-8 = -3(2) + 2c_2$$

$$2c_2 = -8 + 6$$

$$c_2 = \frac{-2}{2} = -1$$

putting the value of c_1 and c_2 in eq (A)

$$y = 2x^{-3} - x^2 + 2\ln x(x^2)$$

$$y = \frac{2}{x^3} - x^2 + 2x^2 \log x \quad \text{Ans}$$

Question # 04

$x^2y'' + 7xy' + 5y = x^5$

$y(0) = 2, \; y'(1) = 2$

Solution:

$$x^2\frac{dy^2}{dx^2} + 7x\frac{dy}{dx} + 5y = x^5$$

$$\Rightarrow \left(x^2\frac{d^2}{dx^2} + 7x\frac{d}{dx} + 5\right)y = x^5 \quad \text{--- (A)}$$

Put $xD = \Theta$

$$\Rightarrow x^2D^2 = \Theta(\Theta - 1) = \Theta^2 - \Theta$$

$x = e^t \Rightarrow \log x = t$ in eq (A)

$$\Rightarrow (\Theta^2 - \Theta + 7\Theta + 5)y = e^{5t}$$

$$\Rightarrow (\Theta^2 + 6\Theta + 5)y = e^{5t}$$

By Quadratic formula

$$\Theta = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Theta = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$$

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$$D = \frac{-6 \pm \sqrt{36-20}}{2}$$

$$= \frac{-6 \pm \sqrt{16}}{2}$$

$$= \frac{-6 \pm \sqrt{4^2}}{2}$$

$$= \frac{\cancel{2}(-3 \pm 2)}{\cancel{2}}$$

$D = -3 \pm 2$, As roots are equal and distinct

$$y_c = C_1 e^{-5t} + C_2 e^{-t}$$

For $y_p = ?$

$$y_p = \frac{1}{D^2 + 6D + 5} e^{5t}$$

$$= \frac{1}{(5)^2 + 6(5) + 5} e^{5t}$$

$$= \frac{1}{60} e^{5t}$$

Now general Solution is

$$y = y_c + y_p$$

$$y = C_1 e^{-5t} + C_2 e^{-t} + \frac{1}{60} e^{5t}$$

$$y = C_1 x^{-5} + C_2 x^{-1} + \frac{1}{60} x^5 \quad \text{--- (B)}$$

$x = 0$ put in the above equation

$$e^0 = 1$$

putting $y(0) = 2$ i.e. $y = 2$ and $x = 0$

$$2 = C_1 (2)^{-5} + C_2 (2)^{-1} + \frac{1}{60} (2)^5$$

$$2 = -32 C_1 - 2 C_2 + \frac{1}{\cancel{60}_{15}} (\cancel{32}^{8})$$

$$2 = -32 C_1 - 2 C_2 + \frac{8}{15}$$

$$2 - \frac{8}{15} = -32 C_1 - 2 C_2 \quad \text{--- (C)}$$

Now differentiate eq C w.r.t x

$$y' = -5C_1 x^{-6} - C_2 x^{-2} + \frac{1}{12} x^4$$

put $y'(1) = 2$, i.e $y' = 2$ and $x = 2$ in above equation

$$2 = -5C_1 (2)^{-6} - C_2 (2)^{-2} + \frac{1}{12}(2)^4$$

$$2 = 320C_1 + 4C_2 + \frac{4}{3}$$

$$\Rightarrow 2 - \frac{4}{3} = 320C_1 + 4C_2$$

$$\Rightarrow \frac{2}{3} = 320C_1 + 4C_2 \quad \text{— D}$$

x ing eq C with 2 and then subtracting eq C from D

$$-\frac{44}{15} = 64C_1 + 4\cancel{C_2}$$

$$+\frac{2}{3} = \pm 320C_1 \pm 4\cancel{C_2}$$

$$\frac{34}{15} = -256C_1$$

$$\boxed{C_1 = 580}$$

Putting the value of C_1 in eq (C)

$$\frac{22}{15} = -32(580) - 2C_2$$

$$\Rightarrow \frac{22}{15} = -18560 - 2(C_2)$$

$$\Rightarrow \frac{22}{15} + 18560 = -2C_2$$

$$\Rightarrow \frac{18560}{-2} = C_2$$

$$\boxed{C_2 = -9280}$$

Now put the value of C_1 and C_2 in eq (B)

$$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x^5$$

$$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5$$

Ans

Question #05

$(x+1)^2 y'' - 3(x+1)y' + 4y = x^2$

Solution:

$$(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2$$

$$\left((x+1)^2\frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4\right) y = x^2$$

$$\Rightarrow [(x+1)^2 D^2 - 3(x+1)D + 4]y = x^2 \quad \text{— (A)}$$

put $\cancel{(x+1)^2 D^2}$

$$(x+1)D = D \Rightarrow (x+1)^2 D^2 = D(D-1)$$
$$= D^2 - D$$

$x = e^t$ in eq (A)

$$\Rightarrow [D^2 - D - 3D + 4]y = e^{2t}$$

$$\Rightarrow [D^2 - 4D + 4]y = e^{2t}$$

$$\Rightarrow [D^2 - 4D + 4]2 = e^{2t}$$

for y_c we find the roots

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$y: D^2 - 4D + 4 = 0$

$D^2 - 2D - 2D + 4 = 0$

$D(D-2) - 2(D-2) = 0$

$D - 2 = 0, \quad D = 2$

$D - 2 = 0, \quad D = 2$

So the roots are real and repeat
The general solution are

$y = (C_1 + C_2 x)^{mx}$

$y = (C_1 + C_2 x)^{2x}$

For $y_p = ?$

$$y_p = \frac{1}{D^2 - 4D + 4}$$

$$y_p = \frac{2e^{2t}}{2D - 4}$$

$2D - 4 = 0$ if we put $D = 2$

$2(2) - 4 = 0$

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we take again derivative

$$Y_p = \frac{x}{x} \cdot e^{2t}$$

$$y = (C_1 + C_2 x)^{2t} + e^{2t}$$ Ans