# Summer Final Paper

Name :- Muhammad Saad

Id # 16097

Subject # Applied Calculus

Department :- BE Civil Engineering

Teacher :- MAM Shomaila Mazhar

Q1 Find PQ wher P is the point in three-dimensional space with coordinates (4, 1, 3) & the point Q with Coordinates (1, 2, 4) ---------- ratio 1:3.

Ans Sol:-

$$P = (4, 1, 3)$$

$$OP = 4i + 1j + 3k$$

or $$OQ = \vec{OQ} - \vec{OP}$$

$$= (i + 2j + 4k) - (4i + 1j + 3k)$$

$$= -3i + 1j + 1k \quad \text{—} (1)$$

Now distance b/w P & Q

$= |PQ|$

$$|PQ| = \sqrt{(-3)^2 + (1)^2 + (1)^2}$$

$$= \sqrt{9 + 1 + 1}$$

$$|PQ| = \sqrt{11} \quad \text{— ②}$$

Let M be the point which divided PQ in ratio 1:3, then by ratio theorem.

* Position vector of $M = \overrightarrow{OM}$

$$= \frac{3(4i + 1j + 3k) + (1)(i + 2j + 4k)}{1 + 3}$$

$$= \frac{12i + 3j + 9k + i + 2j + 4k}{4}$$

$$= \frac{13i + 5j + 13k}{4} \quad \text{— (3)}$$

Hence eq (1), (2) & (3) are the required solution.

Q2 Evaluate $I = \int \frac{4x^3 + 10x + 4}{2x^2 + x} dx$

Sol:-

We will solve it by partial fraction.

$\frac{4x^3 + 10x + 4}{2x^2 + x} dx$ Improper fraction

$$\begin{array}{r|l} & 2x - 1 \\ \hline 2x^2 + x & \cancel{4x^3} + 10x + 4 \\ & \cancel{4x^3} + \quad \pm 2x^2 \\ & \overline{\quad -2x^2 + 10x + 4} \\ & \mp 2x^2 + 10x + 4 \\ & \overline{\quad 11x + 4} \end{array}$$

$= 2x - 1 + \frac{11x + 4}{2x^2 + x} dx$

Integrating

$$I = 2\int (x)\,dx - \int dx + \int \frac{11x+4}{2x^2+x}\,dx$$

$$\frac{2(x)^{1+1}}{1+1} - x + I_1$$

$$= \frac{2x^2}{2} - x + I_1$$

$$= x^2 - x + I_1 \quad \text{--- (1)}$$

$$I_1 = \int \frac{11x+4}{2x^2+x}\,dx$$

$$= \frac{11x+4}{2x^2+x}$$

$$= \frac{11x+4}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1} \quad \text{--- (2)}$$

Xing b.s by $x(2x+1)$

$$11x+4 = A(2x+1) + Bx \quad \text{--- (3)}$$

Scanned with CamScanner

Put $x = 0$ in — 3

$0 + 4 = A(0+1) + 0$

$\boxed{4 = A}$

Put $2x+1 = 0$

$x = -1/2$ in eq 3

$11(-1/2) + 4 = A(0) + B(-1/2)$

$\frac{-11}{2} + 4 = -\frac{1}{2} B$

$\frac{-11+8}{\cancel{2}} = \frac{-B}{\cancel{2}}$

$-3 = -B$

$\boxed{B = 3}$

Put A & B in eq 2

$$\frac{11x+4}{x(2x+1)} = \frac{4}{x} + \frac{3}{2x+1}$$

Integrate

$$\int \frac{11x+4}{x(2x+1)}\,dx = 4\int \frac{1}{x}\,dx + 3\int \frac{1}{2x+1}\,dx$$

$$I_1 = 4\ln(x) + \frac{3}{2}\int \frac{2\,dx}{2x+1}$$

$$= 4\ln x + \frac{3}{2}\ln(2x+1) + c$$

Put in eq ①

$$I = x^2 - x + 4\ln x + \frac{3}{2}\ln(2x+1) + c$$

Q3 (a) $\int_0^2 x^2 e^x \, dx$

Sol:- By product rule

$$x^2 \int_0^2 e^x dx - \int_0^2 \int e^x dx \, \frac{d}{dx} x^2 \, dx$$

$$= x^2 \frac{e^x}{1} \Big|_0^2 - \int_0^2 \frac{e^x}{1} \cdot 2x \, dx$$

$$= x^2 e^x \Big|_0^2 - 2 \int_0^2 x e^x \, dx$$

$$= x^2 e^x \Big|_0^2 - 2\left[x \int_0^2 e^x dx - \int_0^2 \int e^x dx \, \frac{d}{dx} x \cdot dx\right]$$

$$x^2 e^x \Big|_0^2 - 2\left[x \frac{e^x}{1} \Big|_0^2 - \int_0^2 \frac{e^x}{1} \cdot 1 \, dx\right]$$

$$x^2 e^x \Big|_0^2 - 2\left[x e^x \Big|_0^2 - \frac{e^x}{1} \Big|_0^2\right]$$

$$\left[2^2 e^2 - 0(e^0)\right] - 2\left[(2e^2 - 0) - (e^2 - e^0)\right]$$

$= 4e^2 - 2[2e^2 - e^2 + 1]$

$= 4e^2 - 2[e^2 + 1]$

$= 4(2.71)^2 - 2[(2.71)^2 + 1]$

$= 29.38 - 2[7.344 + 1]$

$= 29.38 - 2[8.344]$

$= 12.6884$ Ans.

Q3 b $\int_1^2 \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$

Taking differential on b/s

$d(x)^{1/2} = d(u)$

$\frac{1}{2}(x)^{1/2 - 1} \cdot dx = du$

$\frac{1}{2}(x)^{-1/2} dx = du$

$\frac{1}{2\sqrt{x}} dx = du$

when
$x = 1$
$\sqrt{1} = u$
$u = 1$
when $x = 2$
$u = \sqrt{x}$
$u = \sqrt{2}$

$$\frac{1}{\sqrt{x}}\,dx = 2\,du$$

$$\int_{1}^{\sqrt{2}} \sin u \,(2\,du)$$

$$2\int_{1}^{\sqrt{2}} \sin u \cdot du$$

$$2(-\cos u)\Big|_{1}^{\sqrt{2}}$$

$$-2\cos u\Big|_{1}^{\sqrt{2}}$$

$$-2\left[\cos\sqrt{2} - \cos(1)\right]$$

$$\cancel{-2[}\; -2\cos\sqrt{2} + \cos(1) + C.$$

Ans.

Q4 verify that

$$u(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}} \quad \text{—} \ ①$$

Satify the three- dimensional Laplace's eq.

we know laplace eq in three dimension

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

① w.r.t $x$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-1/2-1} \cdot \frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$= -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2} \cdot (\cancel{2}x+0+0)$$

$$= -x\,(x^2+y^2+z^2)^{-3/2}$$

Diff again.

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial}{\partial x}\left[x\,(x^2+y^2+z^2)^{-3/2}\right]$$

$$= -\left[x\,\partial/_{\partial x}\,(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)\,\partial/_{\partial x}\,x\right]$$

$$= -\left[x\left(-3/2\right)(x^2+y^2+z^2)^{-3/2-1}\,\partial/_{\partial x}\,(x^2+y^2+z^2)\right.$$

$$\left. + (x^2+y^2+z^2)^{-3/2}\cdot 1\right]$$

$$= -\left[\frac{-3x}{\cancel{2}}(x^2+y^2+z^2)^{-3/2-1}\,(\cancel{2}x) + (x^2+y^2+z^2)^{-3/2}\right]$$

Taking $(x^2+y^2+z^2)^{-3/2}$ common.

$$-(x^2+y^2+z^2)^{-3/2}\left[-3x^2\,(x^2+y^2+z^2)^{-1}+1\right]$$

$$-\frac{1}{(x^2+y^2+z^2)^{3/2}}\left[\frac{-3x^2}{x^2+y^2+z^2}+1\right]$$

$$-\frac{1}{(x^2+y^2+z^2)^{3/2}}\left[\frac{-3x^2+x^2+y^2+z^2}{x^2+y^2+z^2}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-1}{(x^2+y^2+z^2)^{3/2+1}}\left[y^2+z^2-2x^2\right] \text{ — (A)}$$

Diff (1) w.r.t y

$$\frac{\partial}{\partial y} u = \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-1/2-1}\,\frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$= -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\,\cancel{2}y$$

$$\frac{\partial u}{\partial y} = -y(x^2+y^2+z^2)^{-3/2}$$

Diff w.r.t y

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial u}{\partial y}\left[y\cdot(x^2+y^2+z^2)^{-3/2}\right]$$

$$= -\left[ y \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial y}(y)\right]$$

$$= -\left[ y\left(-3/2\right)(x^2+y^2+z^2)^{-3/2-1}\cdot\frac{\partial}{\partial y}(x^2+y^2+z^2) + (x^2+y^2+z^2)^{-3/2}\cdot 1\right]$$

$$= -\left[ \frac{-3y}{\cancel{2}}(x^2+y^2+z^2)^{-3/2-1}(\cancel{2}y) + (x^2+y^2+z^2)^{-3/2}\right]$$

$$= \left[-3y^2(x^2+y^2+z^2)^{-3/2-1} + (x^2+y^2+z^2)^{-3/2}\right]$$

Taking $(x^2+y^2+z^2)^{-3/2}$ common

$$-(x^2+y^2+z^2)^{-3/2}\left[-3y^2(x^2+y^2+z^2)^{-1}+1\right]$$

$$-\frac{1}{(x^2+y^2+z^2)^{3/2}}\left[\frac{-3y^2}{x^2+y^2+z^2}+1\right]$$

$$-\frac{1}{(x^2+y^2+z^2)^{3/2}}\left[\frac{-3y^2+x^2+y^2+z^2}{x^2+y^2+z^2}\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-1}{(x^2+y^2+z^2)^{3/2+1}}(x^2-2y^2+z^2) \text{ — B}$$

Diff ① w.r.t $z$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-1/2-1}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}(\cancel{2}z)$$

Diff again

$$\frac{\partial^2 u}{\partial z^2} = -\frac{\partial}{\partial z}\left[z\cdot(x^2+y^2+z^2)^{-3/2}\right]$$

$$= -\left[z\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial z}z\right]$$

$$= -\left[z(-3/2)(x^2+y^2+z^2)^{-3/2-1}\cdot 2z + (x^2+y^2+z^2)^{-3/2}\cdot 1\right]$$

Taking $(x^2+y^2+z^2)$

$$-(x^2+y^2+z^2)^{-3/2}\left[-3z^2(x^2+y^2+z^2)^{-1}+1\right]$$

$$-\frac{1}{(x^2+y^2+z^2)^{3/2}}\left[\frac{-3z^2}{x^2+y^2+z^2}+1\right]$$

$$\frac{\partial^2 u}{\partial z^2}=\frac{-1}{(x^2+y^2+z^2)^{5/2}}(x^2+y^2+2z^2) \quad \text{— C}$$

Adding A B C

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

$$=-\frac{1}{(x^2+y^2+z^2)^{5/2}}(y^2+z^2-2x^2)-\frac{1}{(x^2+y^2+z^2)^{5/2}}(x^2+z^2-2y^2)$$

$$-\frac{1}{(x^2+y^2+z^2)^{5/2}}(x^2+y^2-2z^2)$$

Taking $\frac{-1}{(x^2+y^2+z^2)^{5/2}}$ common

$$= -\frac{1}{(x^2+y^2+z^2)^{5/2}}\left[\cancel{y^2}+\cancel{z^2}-\cancel{2x^2}+\cancel{x^2}-\cancel{2y^2}+\cancel{z^2}+\cancel{x^2}+\cancel{y^2}-\cancel{2x^2}\right]$$

$$= \frac{-0}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

Hence laplace eq verified.