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Differential Equations

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Q#1

i- $w = \sin(x+ct) + \cos(2x+2ct)$

Sol:-

Given $$\frac{d^2w}{dt^2} = c^2\frac{d^2w}{dx^2} \longrightarrow (1)$$

Now

$$\frac{dw}{dt} = \frac{d}{dt}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$= \frac{d}{dt}\left[\sin(x+ct)\right] + \frac{d}{dt}\left[\sin(2x+2ct)\right]$$

$$\frac{dw}{dt} = c\cos(x+ct) - 2c\sin(2x+2ct)$$

Now,

$$\frac{d^2w}{dt^2} = \frac{d}{dt}\left[c\cos(x+ct) - 2c\sin(2x+2ct)\right]$$

$$\frac{d^2w}{dt^2} = -c^2\sin(x+ct) - 4c^2\cos(2x+2ct)$$

Now,

$$\frac{dw}{dx} = \frac{d}{dx}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$\frac{dw}{dx} = \cos(x+ct) - 2\sin(2x+2ct)$$

$$\frac{d^2w}{dx^2} = \frac{d}{dx}\left[\cos(x+ct) - 2\sin(2x+2ct)\right]$$

$$\frac{d^2w}{dx^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

① $\Rightarrow$

$$-c^2\sin(x+ct) - 4c^2\cos(2x+2ct) = c^2[-\sin(x+ct) - 4\cos(2x+2ct)$$

$$-c^2\sin\cancel{(x+ct)} - 4c^2\cos\cancel{(2x+2ct)} = -c^2\sin\cancel{(x+ct)} - 4c^2\cos\cancel{(2x+2ct)}$$

$$0 = 0 \quad (\text{Satisfied})$$

ii) $w = \tan(2x+ct)$

Sol:-

Now,

$$\frac{dw}{dt} = c\cdot\sec^2(2x+ct)$$

&

$$\frac{d^2w}{dt^2} = \frac{d}{dt}[c\cdot\sec^2(2x+ct)]$$

$$= c^2\cdot 2\sec^2(2x+ct)\tan(2x+ct)$$

Now,

$$\frac{dw}{dx} = 2\sec^2(2x+ct)$$

$$\frac{d^2w}{dx^2} = 4\sec^2(2x+ct)\tan(2x+ct)$$

$$4c^2\sec^2(2x+ct)\cancel{\tan}(2x+ct) = 4c^2\sec^2(2\cancel{x}+ct)\tan(2x+ct)$$

$$0 = 0 \quad (\text{satisfied}$$

Q#2

Given function is

$$F(x) = \begin{cases} x \ ; -\pi < x \le 0 \\ 2x \ ; \ 0 \le x \le \pi \end{cases}$$

We have to find the Fourier co-efficient, a_0, a_n & b_n

Now;

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left[0 - \frac{\pi^{\cancel{2}}}{2}\right] + \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^{\cancel{2}}}{\cancel{2}} - 0\right]$$

$$\boxed{a_0 = -\frac{\pi}{2} + \pi = \frac{\pi}{2}} \longrightarrow (1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (x\cos nx\,dx) + \frac{1}{\pi}\int_{0}^{\pi} (2x\cos nx)\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos(0)}{n^2} - \frac{\cos n\pi}{n^2}\right] + \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n + 2(-1)^n - 2}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

$\therefore$

$$a_n = \begin{cases} \dfrac{-2}{\pi n^2} \ ; \text{ if } n \text{ is odd} \\ 0 \ ; \text{ if } n \text{ is even} \end{cases} \longrightarrow (2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx = \frac{1}{\pi}\int x\sin nx\,dx + \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0} + \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$b_n = \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] + \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n}\right] = \frac{-3\cos n\pi}{n} = \frac{3(-1)^{n+1}}{n} \longrightarrow (3)$$

Hence,

the required Fourier series is :

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{nx}$$

Q#3 $\quad y'' - 4y' + 13y = 8\sin 3x \quad , \quad y(0) = 1$ and $y'(0) = 2$

We have to find $y = y_c + y_p$

For y_c the characteristic (auxillary Eqn) Eqn is :

$$m^2 - 4m + 13 = 0$$

$$\Rightarrow m = \frac{4 \pm \sqrt{16-52}}{2} \quad \Rightarrow m = \frac{4+6i}{2}$$

$$\Rightarrow m = 2 \pm 3i \quad ; \quad \alpha = 2 \quad \& \quad \beta = 3$$

So $\quad y_c = e^{2x}\{c_1\cos 3x + c_2\sin 3x\}$

For y_p let,

$$y_p = \text{Imag}\left(\frac{1}{m^2-4m+13}\cdot 8e^{3ix}\right)$$

$$= 8\,\text{Imag}\,\frac{e^{3ix}}{(3i)^2 - 4(3i) + 13}$$

$$= 8\ \text{Imag}\ \frac{e^{3ix}}{-9-12i+13}$$

$$= 8\ \text{Imag}\ \frac{e^{3ix}}{4-12i}$$

$$y_p = 2\ \text{Imag}\ \frac{e^{3ix}}{1-3i} \times \frac{1+3i}{1+3i}$$

$$= 2\ \text{Imag}\cdot \frac{(1+3i)(e^{3ix})}{(1)^2-(3i)^2}$$

$$= 2\ \text{Imag}\cdot \frac{(1+3i)(e^{3ix})}{10}$$

$$y_p = \frac{2}{10}\left[\text{Imag}\cdot(1+3i)(\cos 3x + i \sin 3x)\right]$$

$$y_p = \frac{2}{10}(\sin 3x + 3\cos 3x)$$

$\therefore$ the general solution is

$$y = y_c + y_p$$

$$y = c_1 e^{2x}\cos 3x + c_2 e^{2x}\sin 3x + \frac{2}{10}(\sin 3x + 3\cos 3x)$$

Now,

Using the initial condition $y(0) = 1$

$$y(0) = c_1 e^{(0)}\cos(0) + c_2 e^{(0)}\sin(0) + \frac{2}{10}[\sin(0) + 3\cos(0)]$$

$$1 = c_1(1) + 0 + 0 + \frac{2}{10}[3(1)]$$

$$1 = c_1 + \frac{6}{10}$$

$$c_1 = 1 - \frac{6}{10} \Rightarrow \frac{4}{10}$$

$$= \frac{2}{5}$$

Now,

Using other initial condition $y'(0) = 2$

$\therefore$

$$y' = c_1 2e^{2x}\cos 3x + c_1 e^{2x}(-3\sin 3x) + c_2 2e^{2x}\sin 3x + c_2 e^{2x}(3\cos 3x) + \frac{2}{10}(\cos 3x - 3\sin 3x)$$

~~$y'(0) = c_1 e\ c_1 e$~~

$$y'(0) = c_1 2e^{(0)}\cos(0) + c_1 e^{(0)}[-3\sin(0)] + c_2 2e^{(0)}\sin(0) + c_2 e^{(0)}[3\cos(0)] + \frac{2}{10}[\cos(0) - 3\sin(0)]$$

$$2 = 2c_1 + 0 + 0 + c_2 3(1) + \frac{2}{10}[1 - 3(0)]$$

$$2 = 2c_1 + 3c_2 + \frac{2}{10}$$

$\boxed{\text{taking } c_1 = \frac{2}{5}}$

$$2 = 2\left(\frac{2}{5}\right) + 3\left(\frac{2}{5}\right) + \frac{2}{10}$$

$$\frac{1}{3}\left(2 - \frac{4}{5} - \frac{2}{10}\right) = c_2$$

$$c_2 = \frac{1}{3}\left(\frac{20 - 8 - 2}{10}\right)$$

$$c_2 = \frac{1}{3}$$

$\therefore$ the general solution is

$$y = \frac{2}{5}e^{2x}\cos 3x + \frac{1}{3}e^{2x}\sin 3x + \frac{2}{10}[\sin 3x + 3\cos 3x]$$

Q#4

$$(D^2 - DD')z = \cos x \cos 2y$$

The auxiliary eq is

$$m^2 - m = 0 \quad , m = 0 , m = 1$$

Hence

the complementary function is given by

$$z_c = f_1(y) + f_2(y+x)$$

For the particular integral, we have

$$z_p = \frac{1}{2} \frac{1}{D^2 - DD'} \left[\cos(x+2y) + \cos(x+2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{-1+2} \cos(x+2y) + \frac{1}{-1-2} \cos(x-2y)\right]$$

$$z_p = \frac{1}{2} \cos(x+2y) - \frac{1}{6} \cos(x-2y)$$

Hence,

the complete solution is given by

$$z = f_1(y) + f_2(y+x) + \frac{1}{2} \cos(x+2y) - \frac{1}{6} \cos(x-2y)$$