Differential Equation # Final term Exam

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Q No # 1

part (1) : $w = \sin(x+ct) + \cos(2x+2ct)$

$$\boxed{\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}} \quad \text{———} \quad *$$

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left\{ \sin(x+ct) + \cos(2x+2ct) \right\}$$

$$= \frac{\partial}{\partial t} \sin(x+ct) + \frac{\partial}{\partial t} \cos(2x+2ct)$$

$$\frac{\partial w}{\partial t} = \cos(x+ct) * c - \sin(2x+2ct) * 2c$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) * c^2 - \cos(2x+2ct) * 4c^2 \quad \text{A}$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - \sin(2x+2ct) * 2$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct) \quad \text{B}$$

put A & B in *

$$-c^2 \sin(x+ct) - 4c^2 \cos(2x+2ct) = c^2 \left\{ -\sin(x+ct) - 4\cos(2x+2ct) \right\}$$

$$-c^2 \sin(x+ct) - 4c^2 \cos(2x+2ct) = -c^2 \sin(x+ct) - 4c^2 \cos(2x+2ct)$$

✓ Hence proved!

QNo#01
part (02):

$$w = \tan(2x+ct)$$

Now $$\frac{\partial w}{\partial t} = c\sec^2(2x+ct)$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(c\sec^2(2x+ct)\right)$$

$$= c^2 \cdot 2 \cdot 2\sec^2(2x+ct)\tan(2x+ct)$$

$$\frac{\partial^2 w}{\partial t^2} = 4c^2\sec^2(2x+ct)\tan(2x+ct) \quad \text{(A)}$$

Now $$\frac{\partial w}{\partial x} = 2\sec^2(2x+ct)$$

$$\frac{\partial^2 w}{\partial x^2} = 2 \cdot 2\sec^2(2x+ct)\tan(2x+ct)$$

$$= 4\sec^2(2x+ct)\tan(2x+ct) \quad \text{(B)}$$

putting eq (A) & (B)
in (*)

$$4c^2\sec^2(2x+ct)\tan(2x+ct) = c^2\left(4\sec^2(2x+ct)\tan(2x+ct)\right)$$

$$4c^2\sec^2(2x+ct)\tan(2x+ct) = 4c^2\sec^2(2x+ct)\tan(2x+ct)$$

Hence proved!

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Q NO # 02 Given:

$$F(x) = x, \quad -\pi < x \leq 0$$
$$= 2x, \quad 0 \leq x \leq \pi$$

Solution #

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} 2x\,dx\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\frac{x^2}{2}\Big|_{-\pi}^{0} + x^2\Big|_{0}^{\pi}\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\left(\frac{0}{2} - \frac{(-\pi)^2}{2}\right) + (\pi)^2 - (0)^2\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\frac{-\pi^2}{2} + \pi^2\right\}$$

$$= \frac{-\pi}{2} + \pi$$

$$= \frac{-\pi + 2\pi}{2}$$

x $\boxed{a_0 = \frac{\cancel{3}\pi}{2}}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} x\cos nx\,dx + \int_{0}^{\pi} 2x\cos nx\,dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{x\int_{-\pi}^{0}\cos nx\,dx - \int_{-\pi}^{0} 1\left(\int_{-\pi}^{0}\cos nx\,dx\right)dx + 2x\int_{0}^{\pi}\cos nx\,dx - \int_{0}^{\pi} 2\left(\int_{0}^{\pi}\cos nx\,dx\right)dx\right.$$

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$$a_n = \frac{1}{\pi}\left\{ x\left(\cancel{\frac{\sin nx}{n}}\right)\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\left(\cancel{\frac{\sin nx}{n}}\right)\Big|_{-\pi}^{0} dx + 2x\left(\cancel{\frac{\sin nx}{n}}\right) - \int_{0}^{\pi} 2\left(\cancel{\frac{\sin nx}{n}}\right)\Big|_{0}^{\pi} dx \right.$$

$$= \frac{1}{\pi}(0)$$

$$\boxed{a_n = 0}$$

Now

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left\{ \int_{-\pi}^{0} x\sin nx\, dx + \int_{0}^{\pi} 2x \sin nx\, dx \right\}$$

$$= \frac{1}{\pi}\left\{ x\int_{-\pi}^{0}\sin nx\, dx - \int_{-\pi}^{0} 1\left(\int_{-\pi}^{0}\sin nx\, dx\right)dx + 2x\int_{0}^{\pi}\sin nx - \int_{0}^{\pi} 2\left(\int_{0}^{\pi}\sin nx\, dx\right)dx \right.$$

$$= \frac{1}{\pi}\left\{ x\left(\frac{-\cos nx}{n}\right)\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\frac{-\cos nx}{n}dx + 2x\left(\frac{-\cos nx}{n}\right) - \int_{0}^{\pi}\frac{-\cos nx}{n}dx \right.$$

$$= \frac{1}{\pi}\left\{ \frac{\pi\cos n\pi}{n} - \left(\cancel{\frac{-\sin nx}{n^2}}\Big|_{-\pi}^{0}\right) + \frac{2\pi\cos n\pi}{n} - \left(\cancel{\frac{-\sin nx}{n^2}}\right)\Big|_{0}^{\pi} \right.$$

$$b_n = \frac{1}{\pi}\left\{\frac{\pi\cos n\pi}{n} + \frac{2\pi\cos n\pi}{n}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\frac{\pi}{n} + \frac{2\pi}{n}\right\}\cos n\pi$$

$$= \frac{1}{\pi}\left\{\frac{3\pi}{n}\right\}(-1)^n$$

$$\boxed{b_n = \frac{3(-1)^n}{n}}$$

So $a_0 = \pi/2$

$a_n = 0$

$b_n = \frac{3(-1)^n}{n}$

Therefore,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\cancel{a_n\cos nx} + b_n\sin nx\right)$$

$$= \frac{\pi/2}{2} + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\pi}{4} + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \ldots$$

$b_1 = -3$, $b_2 = \frac{3}{2}$, $b_3 = -1$

(6)

$$f(x) = \frac{\pi}{4} - 3\sin x + \frac{3}{2}\sin 2x - \sin 3x + \cdots$$

Required!

Q No #03

$$y'' - 4y' + 13y = 8\sin 3x \quad \text{— (1)}$$

$$y(0) = 1, \; y'(0) = 2$$

Solution #

The above equation can be write as,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 8\sin 3x$$

$$\left(\frac{d^2}{dx^2} - 4\frac{d}{dx} + 13\right)y = 8\sin 3x$$

$$\left(m^2 - 4m + 13\right)y = 8\sin 3x \quad \boxed{\frac{d}{dx} = D = m} \quad \text{(2)}$$

Associated Homogenous equation,

The Auxilary equation of eq (2) is,

$$m^2 - 4m + 13 = 0$$

Use quadratic formula

$$a = 1, \quad b = -4, \quad c = 13$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

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$$m = \frac{4 \pm \sqrt{-36}}{2}$$

$$m = \frac{4 \pm \sqrt{36i}}{2} \Rightarrow m = \frac{4 \pm 6i}{2}$$

$$m = 2 \pm 3i \Rightarrow m = 2+3i, \ 2-3i$$

$$\boxed{\begin{aligned} m_1 &= 2+3i \\ m_2 &= 2-3i \end{aligned}}$$

So the roots are Complex and repeated.

$$y_c = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) \text{ --- } (*)$$

~~For~~ particular Solution,

$$y_p = A \cos 3x + B \sin 3x \text{ --- } (A)$$

Differentiate w.r.to 'x'

$$y_p' = -3A \sin 3x + 3B \cos 3x$$

Again Differentiate w.r.to x

$$y_p'' = -9A \cos 3x - 9B \sin 3x$$

put y_p' & y_p'' in (1)

$$(-9A\cos 3x - 9B\sin 3x) - 4(-3A\sin 3x + 3B\cos 3x) + 13(A\cos 3x + B\sin 3x) = 8\sin 3x$$

$$-9A\cos 3x - 9B\sin 3x + 12A\sin 3x - 12B\cos 3x + 13A\cos 3x + 13B\sin 3x = 8\sin 3x$$

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$\Rightarrow (4A-12B)\cos 3x + (4B+12A)\sin 3x = 8\sin 3x$

Comparing Co-efficient with eq A

$4A - 12B = 0 \Rightarrow 4A = 12B \Rightarrow \boxed{A = 3B}$ — a

$4B + 12A = 8 \Rightarrow$ b

put a in eq b

$4B + 12(3B) = 8$

$40B = 8 \Rightarrow$

$5B = 1 \Rightarrow \boxed{B = 1/5}$ — c

put in a

$3B = A$

$3(1/5) = A \Rightarrow \boxed{A = 3/5}$ — d

put c & d in A

$y_p = \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x$

The G-sol is

$y = y_c + y_p$

$y = e^{2x}(c_1\cos 3x + c_2\sin 3x) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x$ #

Now, we have to find values of c_1, c_2.

put $x = 0$ & $y = 1$ in eq #

$$1 = e^{2(0)}\left(c_1 \cos 3(0) + c_2 \sin 3(0)\right) + \frac{3}{5}\cos 3(0) + \frac{1}{5}\sin 3(0)$$

$$1 = 1\left(c_1 + 0\right) + \frac{3}{5}\cos 0 + 0$$

$$1 = c_1 + \frac{3}{5} \Rightarrow c_1 = 1 - \frac{3}{5}$$

$$c_1 = \frac{5-3}{5} \Rightarrow \boxed{c_1 = 2/5}$$

Differentiate eq (#) w.r. to x

$$y' = c_1\left(2e^{2x}\cos 3x - 3e^{2x}\sin 3x\right) + c_2\left(2e^{2x}\sin 3x + 3e^{2x}\cos 3x\right)$$
$$- \frac{6}{5}\sin 3x + \frac{3}{5}\cos 3x \quad \text{— } (1')$$

put $y' = 2$ & $x = 0$ in (1)

$$2 = c_1\left(2e^0\cos 0 - 3e^0\sin 0\right) + c_2\left(2e^0\sin 0 + 3e^0\cos 0\right)$$
$$- \frac{6}{5}\sin 0 + \frac{3}{5}\cos 0$$

$$2 = c_1(2) + c_2(3) + \frac{3}{5}$$

$$2c_1 + 3c_2 = 2 - \frac{3}{5} \Rightarrow 2\left(\frac{2}{5}\right) + 3c_2 = \frac{10-3}{5}$$

$$\frac{4}{5} + 3c_2 = \frac{7}{5}$$

$$3c_2 = 3/5 \Rightarrow \boxed{c_2 = 3/15}$$

putting values of c_1 & c_2 in (#)

$$y = e^{2x}\left(\tfrac{2}{5}\cos 3x + \tfrac{3}{15}\sin 3x\right) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x$$

This is the required general Solution.

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Q No# 04 Solve

$$(D^2 - DD')z = \cos x \cos 2y$$

Solution:- We have,

$$(D^2 - DD') = \cos x \cos 2y \quad \text{--- (A)}$$

The Associated Homogeneous Equ.

$$(D^2 - DD')z = 0$$

put $D = m$ where $D = \frac{d}{dx}$

$$m^2 - m = 0$$

$$m(m-1) = 0$$

$$m = 0 \quad , \quad m = 1$$

The roots are real & distinct.

$$z_c = c_1 e^{0x} + c_2 e^{1x}$$

$$z_c = c_1 + c_2 e^{x}$$

For particular solution,
Let

$$z_p = \frac{1}{D^2 - DD'} \cos x \cos 2y$$

$$z_p = \frac{1}{2} \frac{1}{D^2 - DD'} \left[\cos(x-2y) + \cos(x-2y)\right]$$

$$z_p = \frac{1}{2} \left[\frac{1}{D^2 - DD'} [\cos(x-2y)] + \frac{1}{D^2 - DD'} [\cos(x-2y)]\right]$$

using the Integral/Diff

$$z_p = \frac{1}{2} \left[\frac{1}{1+2} \cos(x+2y) + \frac{1}{-1-2} \cos(x-2y)\right]$$

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$$= \frac{1}{2}\left[1\cos(x+2y) - \frac{1}{3}\cos(x-2y)\right]$$

$$z_p = \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Now the general solution,

$$z = z_c + z_p$$

$$\boxed{z = c_1 + c_2 e^{x} + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)}$$

Required Solution.